VLA Memo XXX

Imaging at wide field of view for a plane array.

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Abstract

We consider the array located at a plane. The coordinate system we chose has u, v axes at the array plane, and w axis perpendicular to the array plane. W components of the baselines are equal zero and therefore expressions describing the Fourier transform pairs: IMAGE $\rightleftharpoons$ VISIBILITIES do not include the w component of the vector directed to the current point at the source. As a result this Fourier transform is becoming two dimensional at the whole semi sphere of the sky. This is obviously true for a snapshot observation. The method to connect different time snapshot observations is given.

1 Introduction

The visibility γ as a function of the baseline vector $\vec{D}$ is related with the source brightness distribution $B(\vec{(}\Delta e)$ by the well known expression (see [1] for example)(effect of the array element primary beam is skipped here):

$$\gamma(\vec{D}) = \int B(\vec{\Delta e}) \exp(-j2\pi \vec{D} \cdot \vec{\Delta e}) d\Omega \tag{1}$$

where $\vec{e}$ is the unit vector directed to the point at the sky;
$\vec{e_0}$ is the unit vector directed to the source reference point;
$\vec{\Delta e} = \vec{e} - \vec{e_0}$
$\vec{D}$ the base line vector;
$\cdot$ at the exponent power stands for the scalar product of the vectors.

Consider all antennas of the array located at a plane. Lets chose the following coordinate system for vectors $\vec{D}$ and $\vec{\Delta e}$, $\vec{e}$, $\vec{e_0}$:
$\vec{u}$, $\vec{v}$ are located at the array plane; $\vec{u}$ to south; $\vec{v}$ to east;
$\vec{w}$ is perpendicular to the array plane.

Vector $\vec{e}$ at the chosen coordinate system is determined as:

$$\begin{aligned} \vec{e} = (\quad & \cos(\delta) \cdot \cos(H) \cdot \sin(LAT) - \sin(\delta) \cdot \cos(LAT), \\ & \cos(\delta) \cdot \sin(H), \\ & \cos(Z)) \end{aligned} \tag{2}$$

where Z is zenith angle - the angle between $\vec{e}$ and zenith to the station plane;
δ, H are declination and hour angle of the source;
LAT is the array center latitude.

W component of the $\vec{\Delta e}$ $(\cos(Z) - \cos(Z_0))$ does not contribute to the scalar product in the case of the plane array (our case) because all base line vectors have zero W components.
Therefore the equation 1 can be rewritten as the two dimensional Fourier transform:

$$\gamma(D_u, D_v) = \int B(\Delta e_u, \Delta e_v) \exp(-j2\pi(D_u \Delta e_u + D_v \Delta e_v)\, d\Delta e_u\, d\Delta e_v \tag{3}$$

where $\Delta e_u = \cos(\delta) \cdot \cos(H) \cdot \sin(LAT) - \sin(\delta) \cdot \cos(LAT)$
$- \cos(\delta_0) \cdot \cos(H_0) \cdot \sin(LAT) + \sin(\delta_0) \cdot \cos(LAT)$
$\Delta e_v = \cos(\delta) \cdot \sin(H) - \cos(\delta_0) \cdot \sin(H_0)$

The image can be restored from the set of the visibilities given by equation 3 by the inverse **two** dimensional Fourier transform.
We can make the conclusion here that **the field of view is not limited by the W component problem for a snapshot observation if the array is plane**.

2 How to connect different snapshot observations?

Lets rewrite equation 3 for the first (reference) snapshot session:

$$\gamma(D_u, D_v) = \int B(\Delta e_{u1}, \Delta e_{v1}) \exp(-j2\pi(D_u \Delta e_{u1} + D_v \Delta e_{v1})\, d\Delta e_{u1}\, d\Delta e_{v1} \tag{4}$$

Both the reference ($\vec{e_0}$) and the given ($\vec{e}$) points at the source moves (along different trajectories) at the chosen coordinate system. So the vector $\vec{\Delta e}$ is becoming different for another snapshot session (another time). The vector $\vec{\Delta e_2}$ (the second snapshot session) can be obtained by linear transformation (in some area of δ, H) of the vector $\vec{\Delta e_1}$ (the first snapshot session):

$$\begin{aligned} \vec{\Delta e_2} &= (\Delta e_{u2}, \Delta e_{V2}) \\ &= ((\Delta e_{u1} \cdot a + \Delta e_{v1} \cdot b),\ (\Delta e_{u1} \cdot c + \Delta e_{v1} \cdot d)) \end{aligned} \tag{5}$$

where a, b, c,and d are the elements of the transformation matrix.

Using equation 5 we can rewrite equation 4 given for the first snapshot session to the relevant equation for the second snapshot session:

$$\gamma(D'_u, D'_v) = \int B(\Delta e_{u1}, \Delta e_{v1}) \exp(-j2\pi(D'_u \Delta e_{u1} + D'_v \Delta e_{v1})\, d\Delta e_{u1}\, d\Delta e_{v1} \tag{6}$$

where $D'_u = D_u \cdot a + D_v \cdot c$
$D'_v = D_u \cdot b + D_v \cdot d$

Looking at equations 4 and 6, we see that connection of the second snapshot session provide additional set of visibilities for another base lines D'_u, D'_V and just two dimensional Fourier transform of the source brightness corresponded to the coordinates of the first snapshot session is used. That means the two dimensional inverse Fourier transform can be applied to the two (and clearly more) snapshot sessions visibilities to restore the source brightness distribution corresponded to the coordinates of the first (reference) snapshot session.

3 What range of declinations and right ascensions where all components of the source move relatively the reference point in accordance of the linear matrix for the fixed time?

Combining several snapshot observation using the linear matrix for the baseline conversion is correct only if all components of the source move at the array plane following of the same matrix for the given time or by other words the values a, b, c, and d at the equation 5 are identical for all components of the source (for the fixed time).
This requirement limits the field of view.

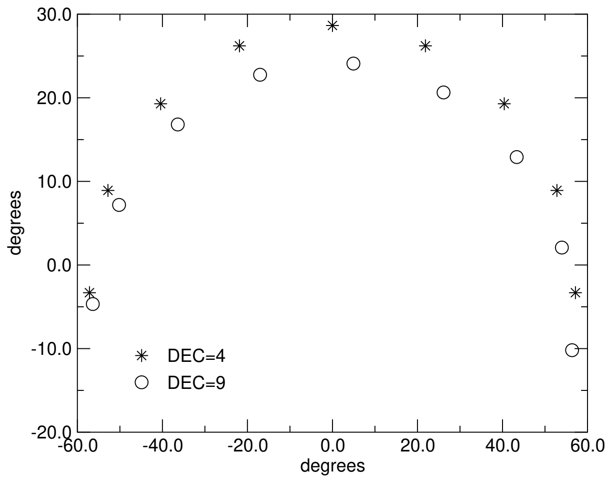

Figure 1: Trajectory of the reference point (declination 4°. labeled by stars) and the point apart in declination and right ascension by 5° (labeled by circles) at the plane zenith angle-azimuth. The array latitude is 34°.

Figure (3) shows the trajectory of the reference point (declination 4° labeled by star) and a source point apart in declination and right ascension by 5° (labeled by circle) at the plane zenith angle-azimuth. The array latitude is 34°. Each circle and star correspond to the hour angle starting with -6 hour going to +6 hour with the step 1.5 hour.

We are looking at the difference vector (between circles and stars). The question is how big is the area on the sky $(\Delta\delta, \Delta\alpha)$ in which the changing of this difference vector from the reference time to the given time is described by the same linear matrix for any source component inside the sky area. This matrix can be different for different time but constant for any sky point inside of the area for the fixed time. We consider a small vicinity of the reference point. So the difference vector can be presented as a linear combination of the vector deviations:

$$\Delta X = \frac{\partial X}{\partial \delta} \cdot \Delta\delta + \frac{\partial X}{\partial H} \cdot \Delta H$$

$$
\begin{aligned}
&= (-\sin\delta\cos H\sin(LAT)\ -\cos\delta\cos(LAT))\cdot\Delta\delta + \\
&\quad (-\cos\delta\sin H\sin(LAT))\cdot\Delta H + \Delta\Delta X \\
\Delta Y &= \frac{\partial Y}{\partial\delta}\cdot\Delta\delta + \frac{\partial Y}{\partial H}\cdot\Delta H \\
&= (-\sin\delta\sin H)\cdot\Delta\delta + (\cos\delta\cos H)\cdot\Delta H\ +\ \Delta\Delta Y
\end{aligned} \tag{7}
$$

where $X = \cos(\delta)\cdot\cos(H)\cdot\sin(LAT) - \sin(\delta)\cdot\cos(LAT)$
$Y = \cos(\delta)\cdot\sin(H)$ (see equation 2)
$\Delta\Delta X, \Delta\Delta Y$ are the relevant deviations of the actual transformation from the linear one.

Lets select the reference time at H=0. Then we can infer from equation (7) the following coordinates of the difference vector at the reference time:

$$
\begin{aligned}
\Delta X_r &= -\cos(\delta - LAT)\cdot\Delta\delta \\
\Delta Y_r &= \cos\delta\cdot\Delta H
\end{aligned} \tag{8}
$$

Substituting equation 8 into equation 7, we can infer the expression related the difference vector coordinates at the given time with the difference vector coordinates at the reference time.

$$
\begin{aligned}
\Delta X &= X2X\cdot\Delta X_r + Y2X\cdot\Delta Y_r + \Delta\Delta X \\
\Delta Y &= X2Y\cdot\Delta X_r + Y2Y\cdot\Delta Y_r + \Delta\Delta Y
\end{aligned} \tag{9}
$$

Where the coefficients of the transformation are determined by the following expression:

$$
\begin{aligned}
X2X &= \frac{\sin\delta\cos H\sin(LAT)\ +\cos\delta\cos(LAT)}{\cos(\delta - LAT)} \\
Y2X &= -\sin H\sin(LAT) \\
X2Y &= \frac{\sin\delta\sin H}{\cos(\delta - LAT)} \\
Y2Y &= \cos H
\end{aligned} \tag{10}
$$

The deviations of the actual transformation from the linear one $(\Delta\Delta X, \Delta\Delta Y)$ can be found through the second derivatives of X and Y:

$$
\begin{aligned}
\Delta\Delta X &= \frac{1}{2}\cdot\left[(-\cos\delta\cos H\sin LAT + \sin\delta\cos LAT)\,\Delta\delta^2 +\ 2(\sin\delta\sin H\sin LAT)\Delta\delta\Delta H + \right. \\
&\quad \left. (-\cos\delta\cos H\sin(LAT))\Delta H^2\ \right] \\
\Delta\Delta Y &= \frac{1}{2}\cdot\left[(-\cos\delta\sin H)\Delta\delta^2\ +\ 2(-\sin\delta\cos H)\,\Delta\delta\Delta H\ +\ (-\cos\delta\sin H)\,\Delta H^2\right]
\end{aligned} \tag{11}
$$

The transformation from the reference time to the given time can be considered as a linear one if the deviations (eq. 11) are less than some portion of the array resolution (say 0.1) $0.1\frac{\lambda}{D}$, where D is the array size. The deviations as determined by equations 11 are the order of $\Delta\delta^2, \Delta H^2$. The values of $\Delta\delta^2, \Delta H^2$ are the half of field of view where the linearity of the transformation is valid. So the field of view is determined by the following expressions:

$$
\Delta\theta_f \simeq 2\sqrt{2}\sqrt{0.1\frac{\lambda}{D}} \sim \sqrt{\frac{\lambda}{D}} \tag{12}
$$

For comparison, the field of view at the usual coordinate system (as given at the page 83 of [1] is three times less.

4 Final expressions for U,V and conversion of the image restored at the horizontal plane to the image at α, δ coordinates

Considering that the reference time is at $H = 0 => t = \alpha$ we can derive (infer) (from equations 6 and 10) the following expressions for U, V.

$$
\begin{aligned}
U &= D_u X2X + D_v X2Y = -D_u \frac{\sin\delta_0 \cos H_0 \sin(LAT) + \cos\delta_0 \cos(LAT)}{\cos(\delta_0 - LAT)} + D_v \frac{\sin\delta_0 \sin H_0}{\cos(\delta_0 - LAT)} \\
V &= D_u Y2X + D_v Y2Y = +D_u \sin H_0 \sin(LAT) + D_v \cos H_0
\end{aligned}
\tag{13}
$$

where $H_0 = t - \alpha_0$ is hour angle of the reference point at the time t
vector $\vec{u}$ is directed to north; vector $\vec{v}$ is directed to east

In particular $U = -D_u,\ \ V = D_v$ for the reference time $(H_0 = 0)$
Conversion of the image restored at the horizontal plane to the image at α, δ coordinates is represented by the simple linear transformation (equation 8):

$$
\begin{aligned}
\Delta\delta &= -\frac{\Delta X_r}{\cos(\delta - LAT)} \\
\Delta\alpha &= -\frac{\Delta Y_r}{\cos\delta}
\end{aligned}
\tag{14}
$$

5 Another way using complanarity of an array.

We can project the source to the UV plane located at the array plane at the fixed time corresponded to the local hour angle equal 0. W axis is perpendicular to this plane. The baselines projected to this coordinate system will be changing during rotation of the Earth. Specifically W projection is equal zero at the small vicinity of the hour angle equaled zero. The question is how big is the time interval where we can consider $W \simeq 0$. The following equations show the dependance of the U,V,W on the U_0, V_0, W_0 and time (hour angle). U_0, V_0, W_0 are the baseline projections to the U,V,W coordinate system corresponded to the array plane at the fixed time corresponded to the local hour angle equal 0.

$$
\begin{aligned}
U &= (-V_0 \sin LAT + W_0 \cos LAT)\sin H + U_0 \cos H \cdot \cos(H) \\
V &= (+V_0 \sin LAT - W_0 \cos LAT)\sin LAT \cos H + U_0 \sin LAT \sin H \\
&+ (V_0 \cos LAT + W_0 \sin LAT)\cos LAT \\
W &= (-V_0 \sin LAT + W_0 \cos LAT)\cos LAT \cos H - U_0 \cos LAT \sin H \\
&+ (V_0 \cos LAT + W_0 \sin LAT)\sin LAT
\end{aligned}
\tag{15}
$$

where U_0, V_0, W_0 are the baseline projections to the U,V axis located at the array plane
and W perpendicular to the plane
LAT is the latitude of array center
H is the local hour angle

For a complanar array, $W_0 = 0$ and W at an arbitrary time (hour angle) can be found from equation 15:

$$
\begin{aligned}
W &= 2\cos LAT \sin\frac{H}{2}\left(V_0 \sin LAT \sin\frac{H}{2} - U_0 \cos\frac{H}{2}\right) \\
&= 2\cos LAT \sin\frac{H}{2}\sqrt{V_0^2 \sin^2 LAT + U_0^2}\ \sin\left(\frac{H}{2} - \varphi\right)
\end{aligned}
\tag{16}
$$

where $\varphi = \arctan(U_0,\ V_0 \sin LAT)$

The sqrt at the equation 16 is approximately equal length of the baseline at the UV plane. So the maximum value of W is equal: $W_{max} = 2\cos LAT \sin\frac{H}{2} B_{max}$. The field of view is inverse proportional to the square root of W_max. So complanarity of the array can give a gain in the field of view which is proportinal to $(2\cos LAT \sin\frac{H}{2})^{-1}$. This gain is especially big for a high latitude of the array, that could be simple predicted. For VLA (LAT = 34°) the gain is more than 2 times if the observation continue less than 1 hour.

6 Are the VLA configurations complanar?

Table 1: Deviation of the VLA antennas off the fitted plane at the zenith direction.

	VLAA	VLAA-E72	VLAB	VLAB-W28	VLAC	VLAC-E18	VLAD	VLAD-N9
Max ΔZ, m	59.3	16.7	14.0	6.5	2.3	1.5	0.78	0.67
RMS Z, m	15.0	6.5	4.4	3.3	0.7	0.4	0.25	0.18
$\Delta\theta_{fp}$, deg	0.8	1.7	2.6	3.4	16.3	28.5	45.6	63.3
$\Delta\theta_f$, deg	0.32	0.32	0.57	0.57	1.04	1.04	1.8	1.8

I have fitted a plane (by least square method) in each of four VLA configurations to see the deviations of actual antenna positions off the plane at the local zenith directions. The result is given at the table 1. I have found that for each of four VLA configuration there is one antenna which deviates of the fitted plain too much. So the table 1 shows the maximum deviation and the rms for all 27 antennas and excluding the most deviated antenna. An array configuration can be considered as a plane one if the phase contribution of the W term exceeds some portion of the turn (say 0.1):

$$\frac{\Delta Z}{\lambda}\frac{\Delta\theta_{fp}}{2} < 0.1 \tag{17}$$

The third line at the table 1 is the field of view estimated from this inequality for $\lambda = 1$m (ΔZ is taken from the RMS' line of the table) The fourth line at the table gives the field of view estimated by equation 12 again for the same wavelength $\lambda = 1$m. It is seen from the table that the fourth line is less than the third one for any VLA configuration. The same conclusion can be made for the shorter wavelength $\lambda = 0.2$m. **Therefore we can consider all VLA configurations being flat!!** at least for the wavelength λ longer than 20cm.

7 Conclusion

Using the coordinate system related with the plane of an array may lead to some advantage in comparison with usually used coordinate system:
1. No limit of the field of view (by W term) for snapshot observation.
2. Simple combine of the different snapshot session staying in two dimensional Fourier transform environment.
3. The snapshot session combine can be carried out at the limited field of view. This field of view is bigger in comparison with standard coordinate system.
4. The facet concept can be used to increase the total field of view. The implementation of this concept can be simpler because all facets are in the plane. Each facet can have a square shape.

References

[1] A.R. Thompson, J.M. Moran, J.W. Swenson, Interferometry and Synthesis in Radio Astronomy, A Wiley-Interscience Publication, 1986