Polarization in Interferometry

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Polarization in interferometry

- Astrophysics of Polarization
- Physics of Polarization
- Antenna Response to Polarization
- Interferometer Response to Polarization
- Polarization Calibration & Observational Strategies
- Polarization Data & Image Analysis
WARNING!

• This is tough stuff. Difficult concepts, hard to explain without complex mathematics.
• I will illustrate the concepts with figures and ‘handwaving’.
• Many good references:
  – Synthesis Imaging II: Lecture 6, also parts of 1, 3, 5, 32
  – Born and Wolf: Principle of Optics, Chapters 1 and 10
  – Rolfs and Wilson: Tools of Radio Astronomy, Chapter 2
  – Thompson, Moran and Swenson: Interferometry and Synthesis in Radio Astronomy, Chapter 4
  – Tinbergen: Astronomical Polarimetry. All Chapters.
• Great care must be taken in studying these – conventions vary between them. 

DON’T PANIC!
Polarization Astrophysics
What is Polarization?

- Electromagnetic field is a vector phenomenon – it has both direction and magnitude.
- From Maxwell’s equations, we know a propagating EM wave (in the far field) has no component in the direction of propagation – it is a transverse wave.

\[ \mathbf{k} \cdot \mathbf{E} = 0 \]

- The characteristics of the transverse component of the electric field, \( \mathbf{E} \), are referred to as the polarization properties. The E-vector follows a (elliptical) helical path as it propagates:
Why Measure Polarization?

• Electromagnetic waves are intrinsically polarized
  – monochromatic waves are fully polarized

• Polarization state of radiation can tell us about:
  – the origin of the radiation
    • intrinsic polarization
  – the medium through which it traverses
    • propagation and scattering effects
  – unfortunately, also about the purity of our optics
    • you may be forced to observe polarization even if you do not want to!
Astrophysical Polarization

• Examples:
  – Processes which generate polarized radiation:
    • Synchrotron emission: Up to ~80% linearly polarized, with no circular polarization. Measurement provides information on strength and orientation of magnetic fields, level of turbulence.
    • Zeeman line splitting: Presence of B-field splits RCP and LCP components of spectral lines by 2.8 Hz/μG. Measurement provides direct measure of B-field.
  – Processes which modify polarization state:
    • Free electron scattering: Induces a linear polarization which can indicate the origin of the scattered radiation.
    • Faraday conversion: Particles in magnetic fields can cause the polarization ellipticity to change, turning a fraction of the linear polarization into circular (possibly seen in cores of AGN)
Example: Radio Galaxy 3C31

- VLA @ 8.4 GHz
- E-vectors
  - along core of jet
  - radial to jet at edge
- Laing (1996)
Example: Radio Galaxy Cygnus A

- VLA @ 8.5 GHz  B-vectors  Perley & Carilli (1996)
Example: Faraday rotation of CygA

- See review of “Cluster Magnetic Fields” by Carilli & Taylor 2002 (ARAA)
Example: Zeeman effect

Zeeman Effect

Atoms and molecules with a net magnetic moment will have their energy levels split in the presence of a magnetic field.

⇒ HI, OH, CN, H₂O

⇒ Detected by observing the frequency shift between right and left circularly polarized emission

⇒ \( V = \text{RCP} - \text{LCP} \propto B_{\text{los}} \)

Energy Levels for HI Ground State

1.42 GHz

Hyperfine transition

\( \Delta E = \mu_B \cdot B \)

\( \Delta v = \frac{g_I \mu_B B}{h} \)

\( \overrightarrow{B} \)

\( \overrightarrow{B}_0 \)

Right Circular Polarization

Linear Polarization

Left Circular polarization

Stokes \( I = \frac{R + L}{2} \)

Stokes \( V = \frac{R - L}{2} \)

Scaled \( dI/dv \)

W51C (2-b) \( B_0 = 2.5 \pm 0.2 \text{ mG} \)
Example: the ISM of M51

- Trace magnetic field structure in galaxies

Neininger (1992)
Scattering

- Anisotropic Scattering induces Linear Polarization
  - electron scattering (e.g. in Cosmic Microwave Background)
  - dust scattering (e.g. in the millimeter-wave spectrum)

Animations from Wayne Hu

Planck predictions – Hu & Dodelson *ARAA* 2002
The Polarization Ellipse

- From Maxwell’s equations $E \cdot B = 0$ (E and B perpendicular)
  - By convention, we consider the time behavior of the E-field in a fixed perpendicular plane, from the point of view of the receiver.

- For a monochromatic wave of frequency $\nu$, we write
  \[ E_x = A_x \cos(2\pi \nu t + \phi_x) \]
  \[ E_y = A_y \cos(2\pi \nu t + \phi_y) \]
  - These two equations describe an ellipse in the (x-y) plane.

- The ellipse is described fully by three parameters:
  - $A_x, A_y$, and the phase difference, $\delta = \phi_y - \phi_x$.

- The wave is elliptically polarized. If the E-vector is:
  - Rotating clockwise, the wave is ‘Left Elliptically Polarized’,
  - Rotating counterclockwise, it is ‘Right Elliptically Polarized’. 
Elliptically Polarized Monochromatic Wave

The simplest description of wave polarization is in a Cartesian coordinate frame.

In general, three parameters are needed to describe the ellipse.

The angle $\alpha = \text{atan}(A_y/A_x)$ is used later …
Polarization Ellipse Ellipticity and P.A.

• A more natural description is in a frame \((\xi,\eta)\), rotated so the \(\xi\)-axis lies along the major axis of the ellipse.

• The three parameters of the ellipse are then:
  \(A_\eta\) : the major axis length
  \(\tan \chi = A_\xi / A_\eta\) : the axial ratio
  \(\Psi\) : the major axis p.a.

\[
\tan 2\Psi = \tan 2\alpha \cos \delta \\
\sin 2\chi = \sin 2\alpha \sin \delta
\]

• The ellipticity \(\chi\) is signed:
  \(\chi > 0 \rightarrow \text{REP}\)
  \(\chi < 0 \rightarrow \text{LEP}\)

\[
\chi = 0,90^\circ \rightarrow \text{Linear (}\delta=0^\circ,180^\circ\text{)}
\]

\[
\chi = \pm 45^\circ \rightarrow \text{Circular (}\delta=\pm 90^\circ\text{)}
\]
Circular Basis

- We can decompose the E-field into a circular basis, rather than a (linear) Cartesian one:

\[ \mathbf{E} = A_R \hat{e}_R + A_L \hat{e}_L \]

- where \( A_R \) and \( A_L \) are the amplitudes of two counter-rotating unit vectors, \( e_R \) (rotating counter-clockwise), and \( e_L \) (clockwise)
- NOTE: R,L are obtained from X,Y by \( \delta = \pm 90^\circ \) phase shift

- It is straightforward to show that:

\[
A_R = \frac{1}{2} \sqrt{A_X^2 + A_Y^2 - 2 A_X A_Y \sin \delta_{XY}}
\]

\[
A_L = \frac{1}{2} \sqrt{A_X^2 + A_Y^2 + 2 A_X A_Y \sin \delta_{XY}}
\]
Circular Basis Example

- The black ellipse can be decomposed into an x-component of amplitude 2, and a y-component of amplitude 1 which lags by $\frac{1}{4}$ turn.
- It can alternatively be decomposed into a counterclockwise rotating vector of length 1.5 (red), and a clockwise rotating vector of length 0.5 (blue).
The Poincare Sphere

- Treat $2\psi$ and $2\chi$ as longitude and latitude on sphere of radius $A=E^2$
Stokes parameters

- Spherical coordinates: radius $I$, axes $Q$, $U$, $V$
  
  - $I = E_X^2 + E_Y^2 = E_R^2 + E_L^2$
  
  - $Q = I \cos 2\chi \cos 2\psi = E_X^2 - E_Y^2 = 2E_R E_L \cos \delta_{RL}$
  
  - $U = I \cos 2\chi \sin 2\psi = 2E_X E_Y \cos \delta_{XY} = 2E_R E_L \sin \delta_{RL}$
  
  - $V = I \sin 2\chi = 2E_X E_Y \sin \delta_{XY} = E_R^2 - E_L^2$

- Only 3 independent parameters:
  
  - wave polarization confined to surface of Poincare sphere
  
  - $I^2 = Q^2 + U^2 + V^2$

- Stokes parameters $I, Q, U, V$
  
  - defined by George Stokes (1852)
  
  - form complete description of wave polarization
  
  - NOTE: above true for 100% polarized monochromatic wave!
**Linear Polarization**

- Linearly Polarized Radiation: $V = 0$
  - Linearly polarized flux:
    
    $$P = \sqrt{Q^2 + U^2}$$
    
  - $Q$ and $U$ define the linear polarization position angle:
    
    $$\tan 2\psi = \frac{U}{Q}$$
    
  - Signs of $Q$ and $U$:

  ![Diagram showing the signs of Q and U](image)
Simple Examples

- If $V = 0$, the wave is linearly polarized. Then,
  - If $U = 0$, and $Q$ positive, then the wave is vertically polarized, $\Psi = 0^\circ$
  - If $U = 0$, and $Q$ negative, the wave is horizontally polarized, $\Psi = 90^\circ$
  - If $Q = 0$, and $U$ positive, the wave is polarized at $\Psi = 45^\circ$
  - If $Q = 0$, and $U$ negative, the wave is polarized at $\Psi = -45^\circ$. 
Illustrative Example: Non-thermal Emission from Jupiter

- Apr 1999 VLA 5 GHz data
- D-config resolution is 14"
- Jupiter emits thermal radiation from atmosphere, plus polarized synchrotron radiation from particles in its magnetic field
- Shown is the I image (intensity) with polarization vectors rotated by 90° (to show B-vectors) and polarized intensity (blue contours)
- The polarization vectors trace Jupiter’s dipole
- Polarized intensity linked to the Io plasma torus
Why Use Stokes Parameters?

• Tradition
• They are scalar quantities, independent of basis XY, RL
• They have units of power (flux density when calibrated)
• They are simply related to actual antenna measurements.
• They easily accommodate the notion of partial polarization of non-monochromatic signals.
• We can (as I will show) make images of the I, Q, U, and V intensities directly from measurements made from an interferometer.
• These I,Q,U, and V images can then be combined to make images of the linear, circular, or elliptical characteristics of the radiation.
Non-Monochromatic Radiation, and Partial Polarization

• Monochromatic radiation is a myth.
• No such entity can exist (although it can be closely approximated).
• In real life, radiation has a finite bandwidth.
• Real astronomical emission processes arise from randomly placed, independently oscillating emitters (electrons).
• We observe the summed electric field, using instruments of finite bandwidth.
• Despite the chaos, polarization still exists, but is not complete – partial polarization is the rule.
• Stokes parameters defined in terms of mean quantities:
Stokes Parameters for Partial Polarization

\[ I = \langle E_x^2 \rangle + \langle E_y^2 \rangle = \langle E_r^2 \rangle + \langle E_l^2 \rangle \]

\[ Q = \langle E_x^2 \rangle - \langle E_y^2 \rangle = 2\langle E_r E_l \cos \delta_{rl} \rangle \]

\[ U = 2\langle E_x E_y \cos \delta_{xy} \rangle = 2\langle E_r E_l \sin \delta_{rl} \rangle \]

\[ V = 2\langle E_x E_y \sin \delta_{xy} \rangle = \langle E_r^2 \rangle - \langle E_l^2 \rangle \]

Note that now, unlike monochromatic radiation, the radiation is not necessarily 100% polarized.

\[ I^2 \geq Q^2 + U^2 + V^2 \]
Summary – Fundamentals

• Monochromatic waves are polarized
• Expressible as 2 orthogonal independent transverse waves
  – elliptical cross-section ➔ polarization ellipse
  – 3 independent parameters
  – choice of basis, e.g. linear or circular
• Poincare sphere convenient representation
  – Stokes parameters I, Q, U, V
  – I intensity; Q,U linear polarization, V circular polarization
• Quasi-monochromatic “waves” in reality
  – can be partially polarized
  – still represented by Stokes parameters
Antenna Polarization
Measuring Polarization on the sky

- Coordinate system dependence:
  - I independent
  - V depends on choice of “handedness”
    - V > 0 for RCP
  - Q,U depend on choice of “North” (plus handedness)
    - Q “points” North, U 45 toward East

- Polarization Angle $\Psi$
  \[ \Psi = \frac{1}{2} \tan^{-1} \left( \frac{U}{Q} \right) \] (North through East)
  - also called the “electric vector position angle” (EVPA)
  - by convention, traces E-field vector (e.g. for synchrotron)
  - B-vector is perpendicular to this
Optics – Cassegrain radio telescope

• Paraboloid illuminated by feedhorn:
Optics – telescope response

- Reflections
  - turn RCP ⇔ LCP
  - E-field (currents) allowed only in plane of surface

“Field distribution” on aperture for E and B planes:

- Cross-polarization at 45°
- No cross-polarization on axes
Example – simulated VLA patterns

- EVLA Memo 58 “Using Grasp8 to Study the VLA Beam” W. Brisken

![Graph showing linear and circular polarization cuts in R & L](image)
Example – measured VLA patterns

• AIPS Memo 86 “Widefield Polarization Correction of VLA Snapshot Images at 1.4 GHz” W. Cotton (1994)
Polarization Receiver Outputs

• To do polarimetry (measure the polarization state of the EM wave), the antenna must have two outputs which respond differently to the incoming elliptically polarized wave.

• It would be most convenient if these two outputs are proportional to either:
  – The two linear orthogonal Cartesian components, \((E_X, E_Y)\) as in ATCA and ALMA
  – The two circular orthogonal components, \((E_R, E_L)\) as in VLA

• Sadly, this is not the case in general.
  – In general, each port is elliptically polarized, with its own polarization ellipse, with its p.a. and ellipticity.

• However, as long as these are different, polarimetry can be done.
Polarizers: Quadrature Hybrids

- We’ve discussed the two bases commonly used to describe polarization.
- It is quite easy to transform signals from one to the other, through a real device known as a ‘quadrature hybrid’.

To transform correctly, the phase shifts must be exactly 0 and 90 for all frequencies, and the amplitudes balanced.
- Real hybrids are imperfect – generate errors (mixing/leaking)
- Other polarizers (e.g. waveguide septum, grids) equivalent
Polarization Interferometry
Four Complex Correlations per Pair

- Two antennas, each with two differently polarized outputs, produce four complex correlations.
- From these four outputs, we want to make four Stokes Images.
Outputs: Polarization Vectors

- Each telescope receiver has two outputs
  - should be orthogonal, close to X,Y or R,L
  - even if single pol output, convenient to consider both possible polarizations (e.g. for leakage)
  - put into vector

\[
\vec{E}(t) = \begin{pmatrix} E_R(t) \\ E_L(t) \end{pmatrix} \quad \text{or} \quad \vec{E}(t) = \begin{pmatrix} E_X(t) \\ E_Y(t) \end{pmatrix}
\]
Correlation products: coherency vector

- Coherency vector: outer product of 2 antenna vectors as averaged by correlator

\[
\vec{v}_{ij} = \left\langle \vec{E}_i \otimes \vec{E}_j^* \right\rangle = \left\langle \begin{pmatrix} E^p_i \\ E^q_i \end{pmatrix} \otimes \begin{pmatrix} E^p_j \\ E^q_j \end{pmatrix} \right\rangle^* = \begin{pmatrix} \langle E_i^p \cdot E_j^p \rangle \\ \langle E_i^p \cdot E_j^q \rangle \\ \langle E_i^q \cdot E_j^p \rangle \\ \langle E_i^q \cdot E_j^q \rangle \end{pmatrix} = \begin{pmatrix} v_{pp} \\ v_{pq} \\ v_{qp} \\ v_{qq} \end{pmatrix}_{ij}
\]

- these are essentially the uncalibrated \textit{visibilities} \(v\)
  - circular products RR, RL, LR, LL
  - linear products XX, XY, YX, YY

- need to include corruptions before and after correlation
Polarization Products: General Case

\[ v_{pq} = \frac{1}{2} G_{pq} \{ I [\cos(\Psi_p - \Psi_q)\cos(\chi_p - \chi_q) + i\sin(\Psi_p - \Psi_q)\sin(\chi_p + \chi_q)] + Q[\cos(\Psi_p + \Psi_q)\cos(\chi_p + \chi_q) + i\sin(\Psi_p + \Psi_q)\sin(\chi_p - \chi_q)] \\
- iU[\cos(\Psi_p + \Psi_q)\sin(\chi_p - \chi_q) + i\sin(\Psi_p + \Psi_q)\cos(\chi_p + \chi_q)] \\
- V[\cos(\Psi_p - \Psi_q)\sin(\chi_p + \chi_q) + i\sin(\Psi_p - \Psi_q)\cos(\chi_p - \chi_q)] \} \]

What are all these symbols?

- \( v_{pq} \) is the complex output from the interferometer, for polarizations \( p \) and \( q \) from antennas 1 and 2, respectively.
- \( \Psi \) and \( \chi \) are the antenna polarization major axis and ellipticity for states \( p \) and \( q \).
- \( I, Q, U, \) and \( V \) are the Stokes Visibilities describing the polarization state of the astronomical signal.
- \( G \) is the gain, which falls out in calibration.

**WE WILL ABSORB FACTOR \( \frac{1}{2} \) INTO GAIN!!!!!!**
Coherency vector and Stokes vector

- Maps (perfect) visibilities to the Stokes vector $s$
- Example: circular polarization (e.g. VLA)
  \[
  \begin{pmatrix}
  v_{RR} \\
  v_{RL} \\
  v_{LR} \\
  v_{LL}
  \end{pmatrix}
  =
  \begin{pmatrix}
  1 & 0 & 0 & 1 \\
  0 & 1 & i & 0 \\
  0 & 1 & -i & 0 \\
  1 & 0 & 0 & -1
  \end{pmatrix}
  \begin{pmatrix}
  I \\
  Q \\
  U \\
  V
  \end{pmatrix}
  =
  \begin{pmatrix}
  I + V \\
  Q + iU \\
  Q - iU \\
  I - V
  \end{pmatrix}
  \]

- Example: linear polarization (e.g. ALMA, ATCA)
  \[
  \begin{pmatrix}
  v_{XX} \\
  v_{XY} \\
  v_{YX} \\
  v_{YY}
  \end{pmatrix}
  =
  \begin{pmatrix}
  1 & 1 & 0 & 0 \\
  0 & 0 & 1 & i \\
  0 & 0 & 1 & -i \\
  1 & -1 & 0 & 0
  \end{pmatrix}
  \begin{pmatrix}
  I \\
  Q \\
  U \\
  V
  \end{pmatrix}
  =
  \begin{pmatrix}
  I + Q \\
  U + iV \\
  U - iV \\
  I - Q
  \end{pmatrix}
  \]
Corruptions: Jones Matrices

- Antenna-based corruptions
  - pre-correlation polarization-dependent effects act as a matrix multiplication. This is the Jones matrix:

\[
\begin{align*}
\mathbf{E}^{\text{out}} &= \mathbf{J} \mathbf{E}^{\text{in}} \\
\mathbf{J} &= \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \\
\mathbf{E} &= \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}
\end{align*}
\]

- form of \( \mathbf{J} \) depends on basis (RL or XY) and effect
  - off-diagonal terms \( J_{12} \) and \( J_{21} \) cause corruption (mixing)
  - total \( \mathbf{J} \) is a string of Jones matrices for each effect

\[
\mathbf{J} = \mathbf{J}_F \mathbf{J}_E \mathbf{J}_D \mathbf{J}_P
\]

- Faraday, polarized beam, leakage, parallactic angle
Parallactic Angle, $P$

- Orientation of sky in telescope’s field of view
  - Constant for equatorial telescopes
  - Varies for alt-az telescopes
  - Rotates the position angle of linearly polarized radiation (R-L phase)

$$J_{RL}^{P} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}; \quad J_{XY}^{P} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$$

- Defined per antenna (often same over array)

$$\phi(t) = \arctan\left(\frac{\cos(l)\sin(h(t))}{\sin(l)\cos(\delta) - \cos(l)\sin(\delta)\cos(h(t))}\right)$$

- $P$ modulation can be used to aid in calibration

$l =$ latitude, $h(t) =$ hour angle, $\delta =$ declination
Visibilities to Stokes on-sky: RL basis

- the (outer) products of the parallactic angle (P) and the Stokes matrices gives

\[
\mathbf{v} = \mathbf{J}_P \mathbf{S} \mathbf{s}
\]

- this matrix maps a sky Stokes vector to the coherence vector representing the four perfect (circular) polarization products:

\[
\begin{pmatrix}
\mathbf{v}^{RR} \\
\mathbf{v}^{RL} \\
\mathbf{v}^{LR} \\
\mathbf{v}^{LL}
\end{pmatrix} =
\begin{pmatrix}
e^{-i(\phi_i-\phi_j)} & 0 & 0 & e^{-i(\phi_i-\phi_j)} \\
0 & e^{-i(\phi_i+\phi_j)} & ie^{-i(\phi_i+\phi_j)} & 0 \\
0 & e^{i(\phi_i+\phi_j)} & -ie^{i(\phi_i+\phi_j)} & 0 \\
e^{i(\phi_i-\phi_j)} & 0 & 0 & -e^{i(\phi_i-\phi_j)}
\end{pmatrix}
\begin{pmatrix}
I \\
Q \\
U \\
V
\end{pmatrix} \xrightarrow{\phi_i=\phi_j=\phi}
\begin{pmatrix}
I + V \\
(Q + iU)e^{-i2\phi} \\
(Q - iU)e^{i2\phi} \\
I - V
\end{pmatrix}
\]

Circular Feeds: linear polarization in cross hands, circular in parallel-hands
Visibilities to Stokes on-sky: XY basis

- we have

\[
\begin{pmatrix}
V^{XX} \\
V^{XY} \\
V^{YX} \\
V^{YY}
\end{pmatrix} = \begin{pmatrix}
\cos(\phi_i - \phi_j) & \cos(\phi_i + \phi_j) & -\sin(\phi_i + \phi_j) & i\sin(\phi_i - \phi_j) \\
-\sin(\phi_i - \phi_j) & \sin(\phi_i + \phi_j) & \cos(\phi_i + \phi_j) & i\cos(\phi_i - \phi_j) \\
\sin(\phi_i - \phi_j) & \sin(\phi_i + \phi_j) & \cos(\phi_i + \phi_j) & -i\cos(\phi_i - \phi_j) \\
\cos(\phi_i - \phi_j) & -\cos(\phi_i + \phi_j) & -\sin(\phi_i + \phi_j) & i\sin(\phi_i - \phi_j)
\end{pmatrix} \begin{pmatrix}
I \\
Q \\
U \\
V
\end{pmatrix}
\]

- and for identical parallactic angles $\phi$ between antennas:

\[
\begin{pmatrix}
V^{XX} \\
V^{XY} \\
V^{YX} \\
V^{YY}
\end{pmatrix}_{\phi_i=\phi_j=\phi} = \begin{pmatrix}
I + Q\cos 2\phi - U\sin 2\phi \\
Q\sin 2\phi + U\cos 2\phi + iV \\
Q\sin 2\phi + U\cos 2\phi - iV \\
I - Q\cos 2\phi + U\sin 2\phi
\end{pmatrix}
\]

Linear Feeds:
linear polarization in all hands, circular only in cross-hands
Basic Interferometry equations

• An interferometer naturally measures the transform of the sky intensity in $uv$-space convolved with aperture
  – cross-correlation of aperture voltage patterns in $uv$-plane
  – its transform on sky is the primary beam $A$ with FWHM $\sim \lambda/D$

$$V(u) = \int d^2x \, A(x - x_p) \, I(x) \, e^{-2\pi i u \cdot (x - x_p)} + n$$

$$= \int d^2v \, \tilde{A}(u - v) \, \tilde{I}(v) \, e^{2\pi i v \cdot x_p} + n$$

– The “tilde” quantities are Fourier transforms, with convention:

$$\tilde{T}(u) = \int d^2x \, e^{-i2\pi u \cdot x} \, T(x) \quad x = (l, m) \leftrightarrow u = (u, v)$$

$$T(x) = \int d^2u \, e^{i2\pi u \cdot x} \, \tilde{T}(u)$$
Polarization Interferometry: Q & U

- Parallel-hand & Cross-hand correlations (circular basis)
  - visibility $k$ (antenna pair $ij$, time, pointing $x$, channel $\nu$, noise $n$):

$$V_{k}^{RR}(u_k) = \int d^2 v \tilde{A}_{k}^{RR}(u_k - v)[\tilde{I}_{\nu}(v) + \tilde{V}_{\nu}(v)]e^{2\pi i v \cdot x_k} + n_k^{RR}$$

$$V_{k}^{RL}(u_k) = \int d^2 v \tilde{A}_{k}^{RL}(u_k - v)[\tilde{Q}_{\nu}(v) + i\tilde{U}_{\nu}(v)]e^{-i2\phi_k} e^{2\pi i v \cdot x_k} + n_k^{RL}$$

$$V_{k}^{LR}(u_k) = \int d^2 v \tilde{A}_{k}^{LR}(u_k - v)[\tilde{Q}_{\nu}(v) - i\tilde{U}_{\nu}(v)]e^{i2\phi_k} e^{2\pi i v \cdot x_k} + n_k^{LR}$$

$$V_{k}^{LL}(u_k) = \int d^2 v \tilde{A}_{k}^{LL}(u_k - v)[\tilde{I}_{\nu}(v) - \tilde{V}_{\nu}(v)]e^{2\pi i v \cdot x_k} + n_k^{LL}$$

- where kernel $A$ is the aperture cross-correlation function, $\phi$ is the parallactic angle, and $\mathbf{Q} + i\mathbf{U} = \mathbf{P}$ is the complex linear polarization

$$\tilde{P}(v) = \tilde{Q}(v) + i\tilde{U}(v) = \tilde{P}(v) e^{i2\phi(v)}$$

- the phase of $\mathbf{P}$ is $\phi$ (the R-L phase difference)
Example: RL basis imaging

- Parenthetical Note:
  - can make a pseudo-I image by gridding RR+LL on the Fourier half-plane and inverting to a real image
  - can make a pseudo-V image by gridding RR-LL on the Fourier half-plane and inverting to real image
  - can make a pseudo-(Q+iU) image by gridding RL to the full Fourier plane (with LR as the conjugate) and inverting to a complex image
  - does not require having full polarization RR,RL,LR,LL for every visibility

- More on imaging ( & deconvolution ) tomorrow!
Polarization Leakage, $D$

- Polarizer is not ideal, so orthogonal polarizations not perfectly isolated
  - Well-designed systems have $d < 1-5\%$ (but some systems $>10\%$ 😞)
  - A geometric property of the antenna, feed & polarizer design
    - frequency dependent (e.g. quarter-wave at center $\nu$)
    - direction dependent (in beam) due to antenna
  - For $R,L$ systems
    - parallel hands affected as $d \cdot Q + d \cdot U$, so only important at high dynamic range (because $Q,U \sim d$, typically)
    - cross-hands affected as $d \cdot I$ so almost always important

\[
\mathbf{J}_{D}^{pq} = \begin{pmatrix}
1 & d^p \\
d^q & 1
\end{pmatrix}
\]

Leakage of q into p (e.g. L into R)
Leakage revisited...

- Primary on-axis effect is “leakage” of one polarization into the measurement of the other (e.g. R ↔ L)
  - but, direction dependence due to polarization beam!
- Customary to factor out on-axis leakage into D and put direction dependence in “beam”
  - example: expand RL basis with on-axis leakage

\[
\hat{V}_{ij}^{RR} = V_{ij}^{RR} + d_i^R V_{ij}^{LR} + d_j^R V_{ij}^{RL} + d_i^R d_j^R V_{ij}^{LL}
\]

\[
\hat{V}_{ij}^{RL} = V_{ij}^{RL} + d_i^R V_{ij}^{LL} + d_j^L V_{ij}^{RR} + d_i^R d_j^L V_{ij}^{LR}
\]

- similarly for XY basis
Example: RL basis leakage

- In full detail:

\[
V_{ij}^{RR} = \int_{\text{sky}} E_{ij}^{RR} (l, m) [(I + V) e^{i(x_i - x_j)} + d_i^R e^{-i(x_i + x_j)} (Q - iU) + d_j^R e^{i(x_i + x_j)} (Q + iU) + d_i^R d_j^R e^{-i(x_i - x_j)} (I - V)] (l, m) e^{-i2\pi(u_{ij}l + v_{ij}m)} dldm
\]

\[
V_{ij}^{RL} = \int_{\text{sky}} E_{ij}^{RL} (l, m) [(Q + iU) e^{i(x_i + x_j)} + d_i^R (I - V) e^{-i(x_i - x_j)} + d_j^R (I + V) e^{i(x_i - x_j)} + d_i^R d_j^R (Q - iU) e^{i(x_i + x_j)}] (l, m) e^{-i2\pi(u_{ij}l + v_{ij}m)} dldm
\]
Example: linearized leakage

- RL basis, keeping only terms linear in \( I, Q \pm iU, d \):

\[
V_{ij}^{RL} = (Q + iU)e^{-i(\phi_i + \phi_j)} + I(d_i^R e^{i(\phi_i - \phi_j)} + d_j^*L e^{-i(\phi_i - \phi_j)})
\]

\[
V_{ij}^{LR} = (Q - iU)e^{i(\phi_i + \phi_j)} - I(d_i^L e^{-i(\phi_i - \phi_j)} + d_j^*R e^{i(\phi_i - \phi_j)})
\]

- Likewise for XY basis, keeping linear in \( I, Q, U, V, d, \sin(\phi_i - \phi_j) \):

\[
V_{ij}^{XY} = Q\sin(\phi_i + \phi_j) + U\cos(\phi_i + \phi_j) + i V + [(d_i^X + d_j^*Y)\cos(\phi_i - \phi_j) - \sin(\phi_i - \phi_j)]I
\]

\[
V_{ij}^{YX} = Q\sin(\phi_i + \phi_j) + U\cos(\phi_i + \phi_j) + i V + [(d_i^Y + d_j^*X)\cos(\phi_i - \phi_j) + \sin(\phi_i - \phi_j)]I
\]

WARNING: Using linear order will limit dynamic range!
Ionospheric Faraday Rotation, $F$

- Birefringency due to magnetic field in ionospheric plasma

$$
J_{FL}^{RL} = \begin{pmatrix}
e^{i\Delta\phi} & 0 \\
0 & e^{-i\Delta\phi}
\end{pmatrix}
$$

$$
J_{FL}^{XY} = \begin{pmatrix}
\cos\Delta\phi & -\sin\Delta\phi \\
\sin\Delta\phi & \cos\Delta\phi
\end{pmatrix}
$$

is direction-dependent

$$
\Delta\phi \approx 0.15^\circ \lambda^2 \int B_\parallel n_e ds
$$

($\lambda$ in cm, $n_e ds$ in $10^{14}$ cm$^{-2}$, $B_\parallel$ in G)

$$
TEC = \int n_e ds \sim 10^{14}$ cm$^{-2}$; \quad B_\parallel \sim 1$G;

$$
\lambda = 20$cm $\rightarrow \Delta\phi \sim 60^\circ$

- also present in ISM, IGM and intrinsic to radio sources!
  - can come from different Faraday depths $\rightarrow$ tomography
Antenna voltage pattern, $E$

- Direction-dependent gain and polarization
  - includes primary beam
    - Fourier transform of cross-correlation of antenna voltage patterns
    - includes polarization asymmetry (squint)

\[
J_{E}^{pq} = \begin{pmatrix}
e_{pp} (l', m') & e_{pq} (l', m') \\
e_{qp} (l', m') & e_{qq} (l', m')
\end{pmatrix}
\]

- includes off-axis cross-polarization (leakage)
  - convenient to reserve $D$ for on-axis leakage
- important in wide-field imaging and mosaicing
  - when sources fill the beam (e.g. low frequency)
Summary – polarization interferometry

• Choice of basis: CP or LP feeds
  – usually a technology consideration

• Follow the signal path
  – ionospheric Faraday rotation $F$ at low frequency
    • direction dependent (and antenna dependent for long baselines)
  – parallactic angle $P$ for coordinate transformation to Stokes
    • antennas can have differing PA (e.g. VLBI)
  – “leakage” $D$ varies with $\nu$ and over beam (mix with $E$)

• Leakage
  – use full (all orders) $D$ solver when possible
  – linear approximation OK for low dynamic range
  – beware when antennas have different parallactic angles
Polarization Calibration & Observation
So you want to make a polarization image...

- Making polarization images
  - follow general rules for imaging
  - image & deconvolve I, Q, U, V planes
  - Q, U, V will be positive and negative
  - V image can often be used as check

- Polarization vector plots
  - EVPA calibrator to set angle (e.g. R-L phase difference)
    - $\Phi = \frac{1}{2} \tan^{-1} \frac{U}{Q}$ for E vectors
  - B vectors $\perp$ E
  - plot E vectors (length given by P)

- Leakage calibration is essential
- See Tutorials on Friday
Strategies for leakage calibration

• Need a bright calibrator! Effects are low level…
  – determine antenna gains independently (mostly from parallel hands)
  – use cross-hands (mostly) to determine leakage
  – do matrix solution to go beyond linear order

• Calibrator is unpolarized
  – leakage directly determined (ratio to $I$ model), but only to an overall complex constant (additive over array)
  – need way to fix phase $\delta_p - \delta_q$ (i.e. R-L phase difference), e.g. using another calibrator with known EVPA

• Calibrator of known (non-zero) linear polarization
  – leakage can be directly determined (for $I,Q,U,V$ model)
  – unknown $p-q$ phase can be determined (from $U/Q$ etc.)
Other strategies

• Calibrator of unknown polarization
  – solve for model IQUV and D simultaneously or iteratively
  – need good parallactic angle coverage to modulate sky and instrumental signals
    • in instrument basis, sky signal modulated by $e^{i2\chi}$
• With a very bright strongly polarized calibrator
  – can solve for leakages and polarization per baseline
  – can solve for leakages using parallel hands!
• With no calibrator
  – hope it averages down over parallactic angle
  – transfer D from a similar observation
    • usually possible for several days, better than nothing!
    • need observations at same frequency
Parallactic Angle Coverage at VLA

- fastest PA swing for source passing through zenith
  - to get good PA coverage in a few hours, need calibrators between declination $+20^\circ$ and $+60^\circ$
Finding polarization calibrators

- Standard sources
  - planets (unpolarized if unresolved)
  - 3C286, 3C48, 3C147 (known IQU, stable)
  - sources monitored (e.g. by VLA)
- other bright sources (bootstrap)

http://www.vla.nrao.edu/astro/calib/polar/
Example: D-term calibration

- D-term calibration effect on RL visibilities (should be Q+iU):

![Graphs showing the effect of D-term calibration on RL visibilities.](image)
Example: D-term calibration

- D-term calibration effect in image plane:
  - Bad D-term solution
  - Good D-term solution
Summary – Observing & Calibration

• Follow normal calibration procedure (previous lecture)

• Need bright calibrator for leakage D calibration
  – best calibrator has strong known polarization
  – unpolarized sources also useful

• Parallactic angle coverage useful
  – necessary for unknown calibrator polarization

• Need to determine unknown $p$-$q$ phase
  – CP feeds need EVPA calibrator for R-L phase
  – if system stable, can transfer from other observations

• Special Issues
  – observing CP difficult with CP feeds
  – wide-field polarization imaging (needed for EVLA & ALMA)