Memo No. 164

Some comments on the W-projection algorithm .

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Abstract

The algorithm W-projection ([paper]) implemented in CASA has been successfully tested using some data. But still the restrictions of this algorithm are not clear. In this memo we point out to the possible problems at the case of poor UV coverage for some W layers. We have evaluated analytically the convolution function which is used at the W-projection algorithm to project (recalculate) the measured visibilities (corrupted by the W term) to the visibilities corresponded to the ideal observations (W=0). The found analytical equation for the convolution function allows to find a simple expression for its width for arbitrary field of view and W. This width is much wider in some cases than the upper limit (70x70 pixels) cited at the [paper] and therefore the computation time may not be as small as is cited there. We suggest multiplication by the $\exp(j\pi Wl^2)$ at the image plane (instead of the convolution at the UV plane). Such a way to correct the W-term may give the gain in the computation time.

1 Introduction

At most cases of the aperture synthesis, the both brightness distribution B(l,m) and the measured visibilities Vis(u,v) are two dimensional functions related through the two dimensional Fourier transform. The modern algorithms of deconvolution (CLEAN, Method maximum entropy) have been successfully applied to restore the image of the source using a two dimensional set of the measured visibilities (U,V). But actually the set of measured visibilities is three dimensional (U,V,W). The effect of the W term can be ignored if the field of view is rather narrow, but can not be ignored for the rather wide field of view. Many different ideas have been considered to remove effect of the W term. One of them is idea of projection (recalculation) the measured visibility with $W \neq 0$ into the corresponding visibilities with $W = 0$. Having carried out such a projection the standard technique of two dimensional imaging can be used. R.H. Frater ([Frater]) was the first to discuss this idea. In particular, he considered the concept of W-layers and different presentations of the convolving function. Recently this algorithm was coded in CASA and got the name W-projection. The W-projection program has been successfully tested several times. In this memo I analyze the possible problem of the W-projection at the case of poor UV coverage. For this purpose I have evaluated the convolution function analytically. I have found that requirement to the width of the convolution function can be more severe (than at [paper]) for the 327 MHz and especially 75 MHz cases.

2 The projection of the measured visibilities at $W \neq 0$ to the desired visibilities at $W = 0$.

The visibility for $W = 0$ ($Vis_{W=0}(U)$) for one dimensional case can be written as the following integral.

$$Vis_{W=0}(U) = \int \underbrace{B(l)\exp(j\pi Wl^2)}_{f_1(l)} \underbrace{\exp(-j\pi Wl^2)}_{f_2(l)} \exp(j2\pi Ul)\, dl \tag{1}$$

where l is the coordinate at the source picture plane;
U is the U components of the baseline vector;
B(l) is the source brightness distribution.

The equation (1) is the Fourier transform of the production of the two functions f_1 and f_2, that is equal to convolution of the Fourier transform of f_1 and f_2. The Fourier transform of f_1 is $Vis_{W\neq 0}$
Therefore the equation (1) can be rewritten as:

$$Vis_{W=0}(U) = Vis_{W\neq 0} \diamond G_w = \int Vis_{W\neq 0}(U_1) \ * G_W(U - U_1)\, dU_1 \tag{2}$$

where the sign $\diamond$ stands for convolution;
$G_W(U)$ is the Fourier transform of the function f_2;

The relation 2 is the mathematical basis for recalculation (projection) of the measured visibilities $Vis_{W\neq 0}$ to the desired visibilities $Vis_{W=0}$
But the relation 2 is correct only if $Vis_{W\neq 0}(U_1)$ is known everywhere in $(-\infty, +\infty)$. In reality only few points of $Vis_{W\neq 0}(U_1)$ can be available at the given W-layer especially at the large W-layers.

Still lets try to estimate the effect of the poor UV coverage at the W-layer on the quality of recalculation (projection) of the measured visibilities $Vis_{W\neq 0}$ into the desirable visibilities with $W = 0$ - $Vis_{W=0}$. Write down the Fourier transform of the equation 2 (dirty map DM), taking into account that the Fourier transform of the convolution of two functions is equal to product of the relevant Fourier transforms, and that the set of measured visibilities can be represent as a product of the indefinite set of the measurement and the SHA function (set of delta functions at the locations of the measured visibilities at the UV plane).

$$\begin{aligned} DM_W(l_1) &= \mathcal{F}(Vis_{W=0}) = [\mathcal{F}(SHA_W) \diamond \mathcal{F}(Vis_{W\neq 0})] * \mathcal{F}G_w \\ &= [DB_W(l) \diamond (B(l) * \exp(-j\pi Wl^2))] * \exp(j\pi Wl_1^2) \\ &= \left[\int DB_W(l_1 - l) * B(l) * \exp(-j\pi Wl^2)\, dl\right] * \exp(j\pi Wl_1^2) \\ &= \int B(l) * DB_W(l_1 - l) * \exp(j\pi W(l_1^2 - l^2)\, dl \end{aligned} \tag{3}$$

The equation 3 confirms that effect of the W term is canceled completely only at the case of the full UV coverage (The dirty beam DM is indefinitely narrow). If the main lobe of the dirty beam DB_W is narrow, then $l_1^2 - l^2 \approx 2(l_1 - l) * l$ and therefore the residual phase corruption when the algorithm W-projection is applied is described by the following equation:

$$\Delta\varphi_{yeswp} \simeq 2\pi W_\lambda \cdot l\Delta l \simeq 2\pi \frac{W}{\lambda} \cdot \frac{\lambda}{d} \cdot \frac{\lambda}{U_{max}(W)} \simeq 2\pi \frac{W}{d} \cdot \frac{\lambda}{U_{max}(W)} \tag{4}$$

where d is the diameter of the array antenna;
W_λ is W in λ at the W- layer;
$U_{max}(W)$ is maximum of U component of the baselines at the W layer;

For comparison, the phase corruption without any correction of the W term is determined by the following equation:

$$\Delta\varphi_{nowp} \simeq \pi\frac{W}{d}\cdot\frac{\lambda}{d} \tag{5}$$

Dividing equation 5 by equation 4, we get equation for the gain of the W-projection algorithm in the suppression of the phase corruption:

$$\frac{\Delta\varphi_{nowp}}{\Delta\varphi_{yeswp}} \simeq \frac{U_{max}(W)}{2d} \tag{6}$$

Looking at this equation everyone can be impressed by the suppression of the phase corruption provided by W-projection algorithm. But there are two concerns:
1. The value of the residual phase corruption when the algorithm W-projection is applied (equation 4) must be small. This is not true at all cases. For example, if $W \sim U_{max}$, then $\Delta\varphi_{yeswp} \simeq 1rad$ (for d=25m, $\lambda = 4m$), what is not so small.
2.The effect of big side lobes at the Dirty beam for some W layers with bad UV coverage can prevent using concept of narrow width of the Dirty beam.

3 W-term correction at the image plane

We can arrive exactly to the same result (equation 3), if carry out the Fourier transform (using FFT) of the observed visibilities at the given W-layer (we'll get $[DB_W(l) \diamond (B(l) * \exp(-j\pi Wl^2))]$) and then correct W term effect at the image plane multiplying the result by the $\exp(j\pi Wl^2)$.
Compare the computational time of the two algorithms:
W-projection requires
$N_{uv} \cdot N_{wc}$ calculations for each W=layer;
The correction at the image plane requires
$N_{lm} \cdot \lg N_{lm}$ calculations for each W=layer;
where: N_{uv}, N_{lm} are the number of pixels at UV and lm planes;
N_{wc} is number of pixels at the width of the convolution function;
Consider $N_{uv} = N_{lm} = N$
The following inequality $N_{wc} \gg \lg(N)$ is practically correct always!!
This proves that the correction of the W-term at the image plane may have an advantage at the computation time in comparison with W-projection algorithm

4 Analytical evaluation of the convolution function

Following the conclusion of the section 1, we can say that the measured visibilities corrupted by the complex exponent with the W term, can be recalculated (projected) to the desired visibilities (without effect of the W term), convolving the measured visibilities with the convolving function $G_W(U)$ which is the Fourier transform of the function $f_2(l) = exp(j\pi Wl^2)$. So the convolving function $G_W(U)$ is determined by the following integral:

$$G_W(U) = \int_{-L}^{L} \exp(j\pi Wl^2)\ \exp(j2\pi Ul)\, dl \tag{7}$$

where W is value of the visibility's W at the W layer, in wavelength;
U is the U components of the baseline vector, in wavelength;
2L is the full field of view, in radians.

This integral can be easily converted to the following integral:

$$G_W(U) = \exp\left(-j\pi\frac{U^2}{W}\right)\int_{-L}^{L}\exp\left(j\pi W\left(l+\frac{U}{W}\right)^2\right)\,dl \tag{8}$$

Now, having changed the variable of the integral $x = \left(l+\frac{U}{W}\right)\sqrt{2W}$, we can come to the following integral:

$$G_W(U) = \exp\left(-j\pi\frac{U^2}{W}\right)\frac{1}{\sqrt{2W}}\int_{x_1}^{x_2}\exp\left(j\frac{\pi}{2}x^2\right)\,dx \tag{9}$$

where

$$\begin{aligned} x_1 &= \left(-L+\frac{U}{W}\right)\sqrt{2W} \\ x_2 &= \left(+L+\frac{U}{W}\right)\sqrt{2W} \end{aligned} \tag{10}$$

Finally the amplitude and the phase of the convolving function $G_W(U)$ can be derived as:

$$|G_W(U)| = \frac{1}{\sqrt{2W}}\sqrt{\left[C(x_2)-C(x_1)\right]^2 + \left[S(x_2)-S(x_1)\right]^2} \tag{11}$$

$$PHAS\left(G_W(U)\right) = \left(-\pi\frac{U^2}{W}\right) + \arctan\left(\frac{S(x_2)-S(x_1)}{C(x_2)-C(x_1)}\right) \tag{12}$$

where $C(x) = \int_0^x \cos\frac{\pi}{2}t^2\,dt$;
$S(x) = \int_0^x \sin\frac{\pi}{2}t^2\,dt$;
are Frenel's integrals ([Janke])

The Frenel's integrals have the following property ([Janke]):
1. Both of them are odd functions;
2. Both of them are equal zero for zero argument;
3. Both of them are going to 0.5 when the argument goes to infinity.

Based on these properties, and considering $L\sqrt{2W} \gg 1$ (what is right when the W problem exists) we arrive to the conclusion that $|G_W(0)| = \frac{1}{\sqrt{2W}}\sqrt{2}$ and therefore the amplitude of the normalized convolving function is equal:

$$\left|\frac{G_W(U)}{G_W(0)}\right| = \frac{1}{\sqrt{2}}\sqrt{\left[C(x_2)-C(x_1)\right]^2 + \left[S(x_2)-S(x_1)\right]^2} \tag{13}$$

Both x_1 and x_2 are positive and very large for large U because of the factor $\sqrt{2W}$ (equation 10) which is big when the W problem exists. So $C(x_2) \simeq 0.5, C(x_1) \simeq 0.5, S(x_2) \simeq 0.5, S(x_1) \simeq 0.5$ and therefore the normalized convolution function (equation 13) is zero for large U.
For the small U x_1 is large negative, but x_2 is large positive (equation 10) and therefore the normalized convolution function (equation 13) is near 1 for small U
If $U = W*L$ then $x_1 = 0$ and $x_2 \gg 1$ (equation 10) and $C(x_1) = S(x_1) = 0$ and $C(x_2) \simeq S(x_2) = 0.5$.
Therefore, if $U = W*L$ then the normalized convolution function (equation 13) is equal 0.5!! Taking into account that the pixel size at U should be equal the reciprocal of double size of the image ($\frac{1}{4L}$) we arrive to the very simple equation for the full width (at the level 0.5) of the convolution function at pixels, if no oversampling:

$$2U_{pix0.5} = 8 * L^2 * W \tag{14}$$

where L is half of the field of view, in radians;
W is in wavelength;
$2U_{pix0.5}$ the full width of the convolution function
at the level 0.5, in pixels (without oversampling).

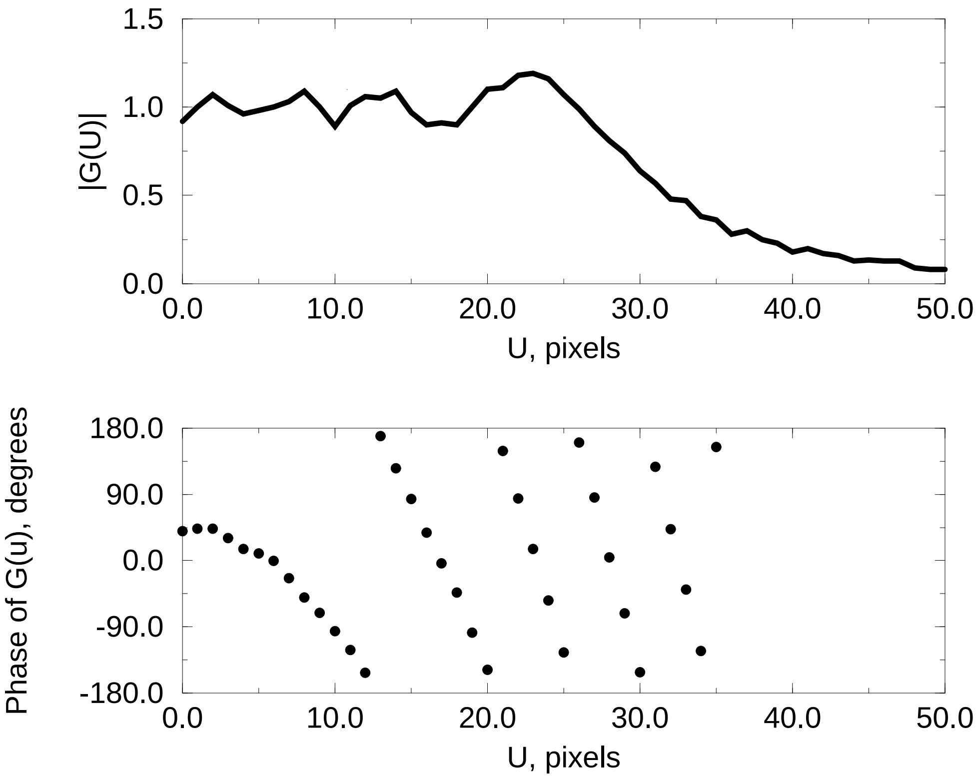

Figure 1: Amplitude and phase of the convolution function. VLA-C, $\lambda = 4m, W = 800\lambda$

If the field of view is fixed, then the width of the convolution function **varies directly with frequency**. But if the field of view is determined by the primary beam width, then the width of the convolution function **varies inversely with frequency**.
The results of calculation of the full width of the convolution function using the equation 14 are given at the table 1 for the four VLA configurations, wavelength 4m, and W=W_{max}. The full size of the image is taken as the full size of the VLA 25m dish beam pattern (level 0.5) at 4band. This size (2L= 700'= 0.204rad) is picked up from the memo: "Low frequency observation and data reduction with VLA" at http:/lwa.nrl.navy.mil/tutorial.

Table 1: The full width of the convolution function at the level 0.5 for the four VLA configurations at $\lambda = 4m$;

VLA, config	D	C	B	A
W_{max}/λ	250	750	2500	7500
$2U_{pix0.5}$, pixels	21	62	210	625

The paper [paper] says: "The support of the w-dependent gridding function grows with both W and the field of view (*equation 14 confirms and specifies this relation*), **typically up to a largest value of about 70 by 70..**" Looking at the table 1, we can say that the paper statement (**70 by 70..**) is right for the $\lambda = 4m$ only for up to VLA-C configuration and the configurations B and A require much more size. This size will be required 10 times bigger for A configuration (100! times bigger at the 2 dimensional case). If the oversampling is desirable then instead of 512 by 512 (as it is in [paper]) it can be 5000 by 5000!!
As an example I have calculated the amplitude (equation 13) and phase (equation 12) of the normalized

convolving function for VLA-C configuration and wavelength 4m. The result is plotted at the Fig 1. The phase is badly determined ($\frac{0}{0}$) when the amplitude drops to < 0.5. So the phase plot stops after U=35.

5 Conclusion

1. The convolution function (both amplitude and phase) is evaluated analytically. The expression for width of the convolution function is found as a very simple equation.
2. Based on the found expression for the width of the convolution function, we found that the requirement for the support of the W-dependent gridding function can be more severe than it is written at the [paper] (70x70 pixels). For example this full width can need be $\sim$ 600x600 pixels at the wavelength 4m for VLA-A configuration. If oversampling is required, then it can be grow up to 5000x5000.
3. So the W-projection algorithm may meet a problem at the longer VLA configurations (VLA-A, VLA-B)
4. The projection of the given visibility ($W \neq 0$) to the desired visibility ($W = 0$) may give a wrong result at the W layers with bad UV coverage.
5. So the W-projection algorithm will not work for a snapshot observation when the UV coverage is especially bad.
6. The computing time (in the [paper]) is compared with the so called facet in UV plane algorithm. **This algorithm is not known to anyone!**
The AIPS task IMAGR (especially facets at the tangent plane (AIPS memo 113)) can give better computing time than W-projection?
7.Correction of the W-term can be carried out at the image plane: the Fourier transform of the observed visibilities at the given W-layer with following multiplication by the $\exp(j\pi W(l^2+m^2))$. This leads to the same equation (3) for the dirty map as W-projection algorithm gives. **The correction at the image plane may have the advantage of the W-projection algorithm in the computing time?**

References

[paper] T.J. Cornwell K. Golap, and S. Bhatnagar, The non-complanar baselines effect in radio interferometry: The W-projection algorithm, IEEE Journal of Selected Topics in Signal Processing v2, N5, Oct 2009 pp 647-657.

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