Fundamentals of Radio Interferometry

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Topics

- Some Definitions: Intensity, Flux Density, etc.
- The Role of the 'Sensor' (a.k.a. 'Antenna')
- Key Properties of Antennas
- The Basic Interferometer

Spectral Flux Density and Brightness

- **Our Goal**: To measure the characteristics of celestial emission from a given direction **s**, at a given frequency ν, at a given time **t**.
  - In other words: We want a map, or image, of the emission.
- **Terminology/Definitions**: The quantity we seek is called the brightness (or specific intensity): It is denoted here by $I(\mathbf{s},\nu,t)$, and expressed in units of: watt/(m^2 Hz ster).
- It is the power received, per unit solid angle $d\Omega$ from direction **s**, per unit collecting area dA, per unit frequency $d\nu$ at frequency ν.
- Do not confuse I with Spectral Flux Density, S -- the integral of the brightness over a given solid angle:

$$S = \int I(\mathbf{s},\nu,t)\,d\Omega$$

- The units of S are: watt/(m^2 Hz)
- Note: 1 Jy = 10^{-26} watt/(m^2 Hz).

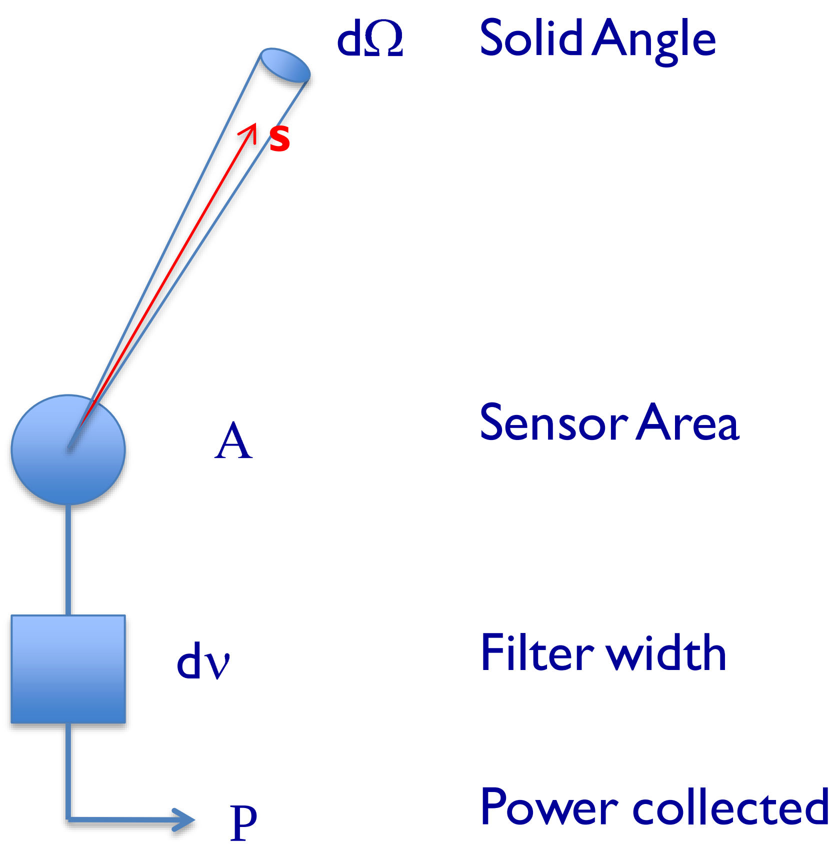
Intensity and Power.

- Imagine a distant source of emission, described by brightness $I(\nu,\mathbf{s})$ where **s** is a unit direction vector.
- Power from this emission is intercepted by a collector (`sensor') of area $A(\nu,\mathbf{s})$.
- The power, dP (watts) from a small solid angle $d\Omega$, within a small frequency window $d\nu$, is

$$dP = I(\nu,\mathbf{s})A(\nu,\mathbf{s})\,d\nu\,d\Omega$$

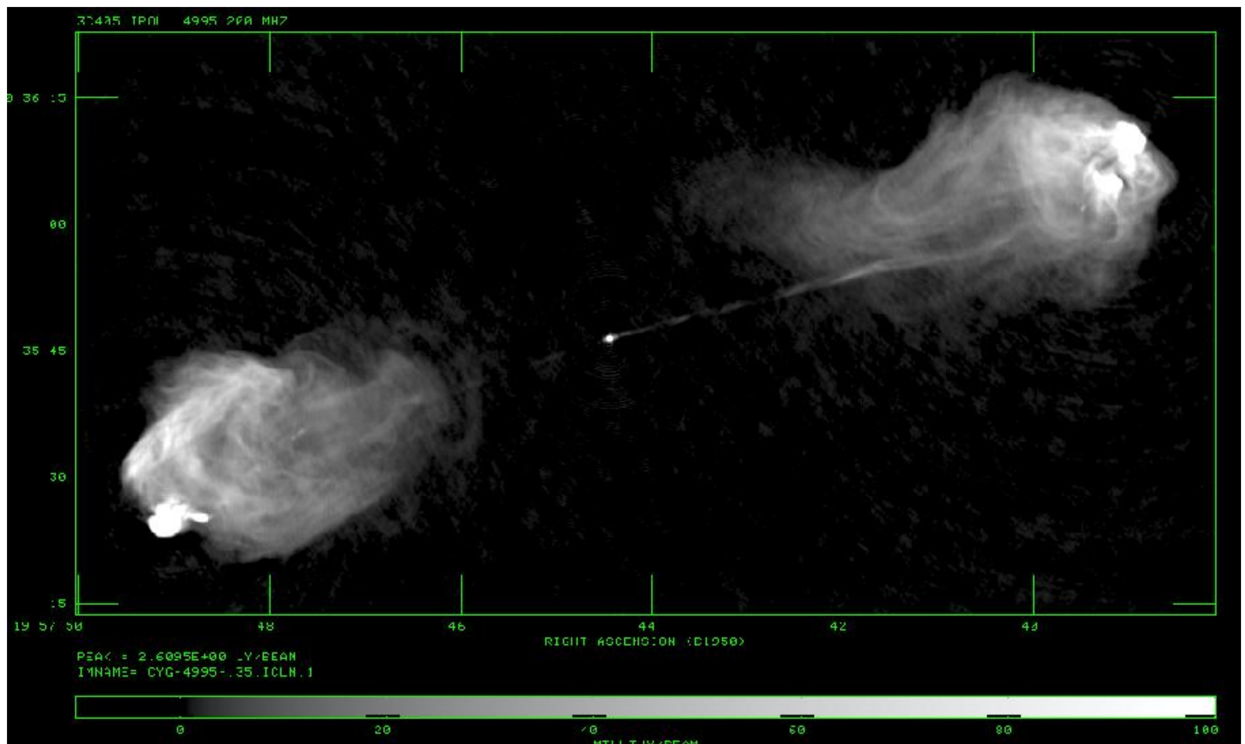
- The total power received is an integral over frequency and angle, accounting for variations in the responses.

$$P = \iint I(\nu,\mathbf{s})A(\nu,\mathbf{s})\,d\nu\,d\Omega$$



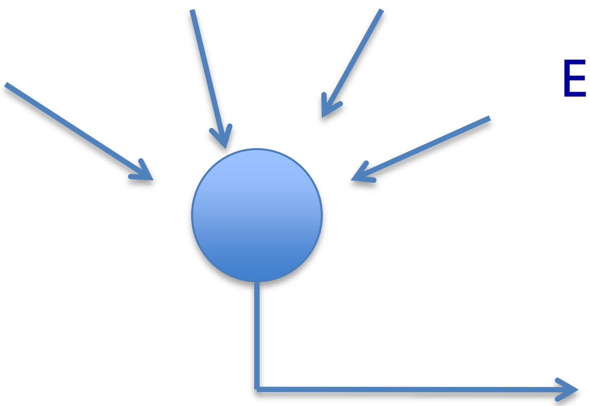
An Example – Cygnus A

- I show below an image of Cygnus A at a frequency of 4995 MHz.
- The units of the brightness are Jy/beam, with 1 beam = 0.16 arcsec^2
- The peak is 2.6 Jy/beam, which equates to 6.5 x 10^{-15} watt/(m^2 Hz ster)
- The flux density of the source is 370 Jy = 3.7 x 10^{-24} watt/(m^2 Hz)



The Role of the Sensor

- Coherent interferometry is based on the ability to correlate the electric fields measured at spatially separated locations.
- To do this (without mirrors) requires conversion of the electric field $E(\mathbf{r},\nu,t)$ at some place to a voltage $V(\nu,t)$ which can be conveyed to a central location for processing.
- For our purpose, the sensor (a.k.a. 'antenna') is simply a device which senses the electric field at some place and converts this to a voltage which faithfully retains the amplitudes and phases of the electric fields.



EM waves in

Voltage out (preserving amplitude and phase of all input fields)

Ideal vs. Real Sensor (Antennas)

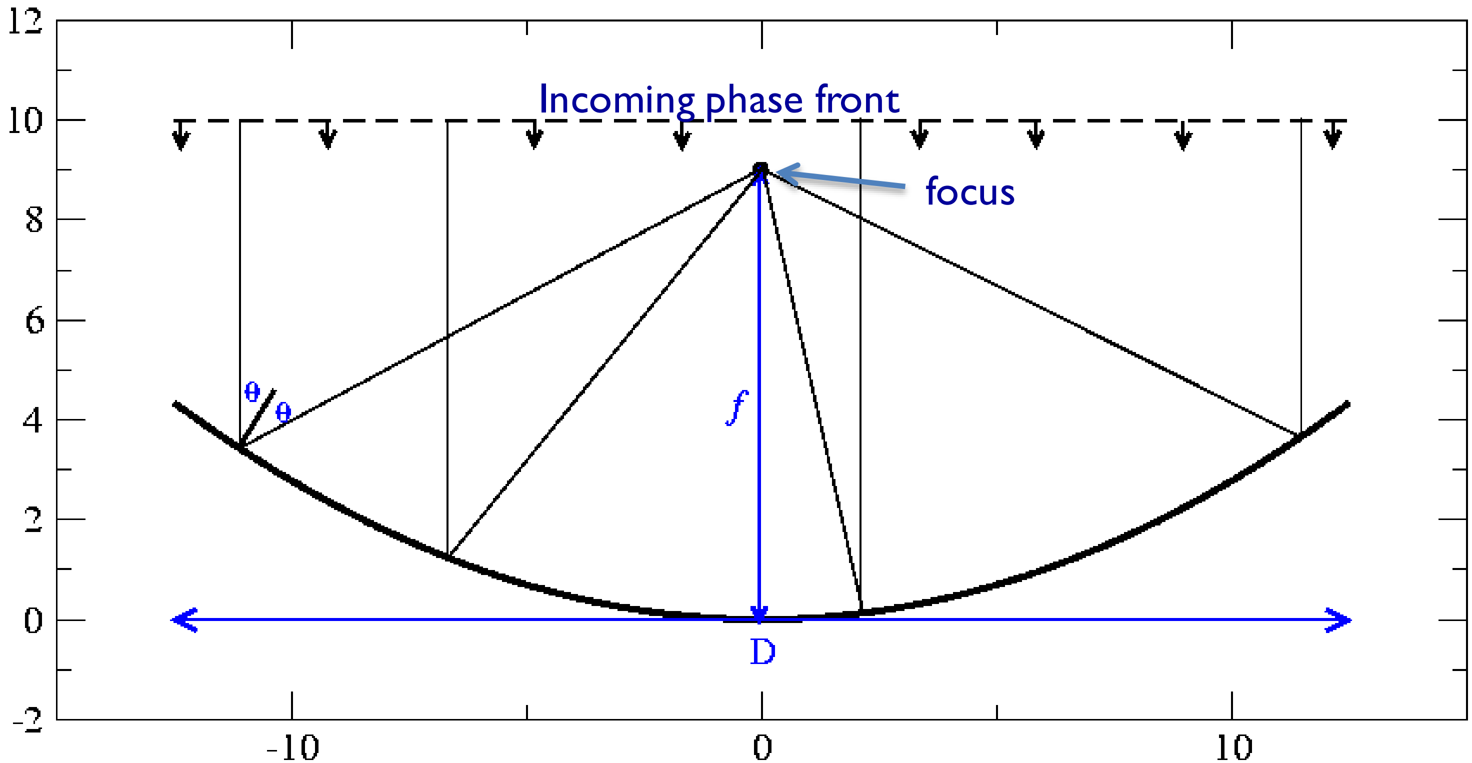
- An ideal antenna converts only the fields of interest, and adds no additional signal of its own.
- One can imagine two kinds of ideal sensors:
  - An 'all-sky' sensor: All incoming electric fields, from all directions, are converted with no direction modulation.
  - The 'limited-field-of-view' sensor: Only the fields from a given direction and solid angle (field of view) are collected and conveyed.
- Sadly – neither of these is possible.
- Nor is it possible to build a sensor which does not add in additional power.
  - For even the best antennas, at cm-wave frequencies, a 1 Jy source contributes less than 0.5% of the antenna output power!

Antennas – the Single Dish

- Antennas designs span a wide range – from simple elements with nearly isotropic responses, to major mechanical structures designed for high gain and angular resolution.
- The most common antenna is a parabolic reflector – a 'single dish'.
- Understanding how it works will help in our later discussion of interferometry.
- There are four critical characteristics of sensors (antennas):
  1. A directional gain ('main beam')
  2. An angular resolution given by: $\theta \sim \lambda/D$.
  3. The presence of 'sidelobes' – finite response at angles away from the main beam.
- A basic understanding of the origin of these characteristics will aid in understanding the functioning of an interferometer.

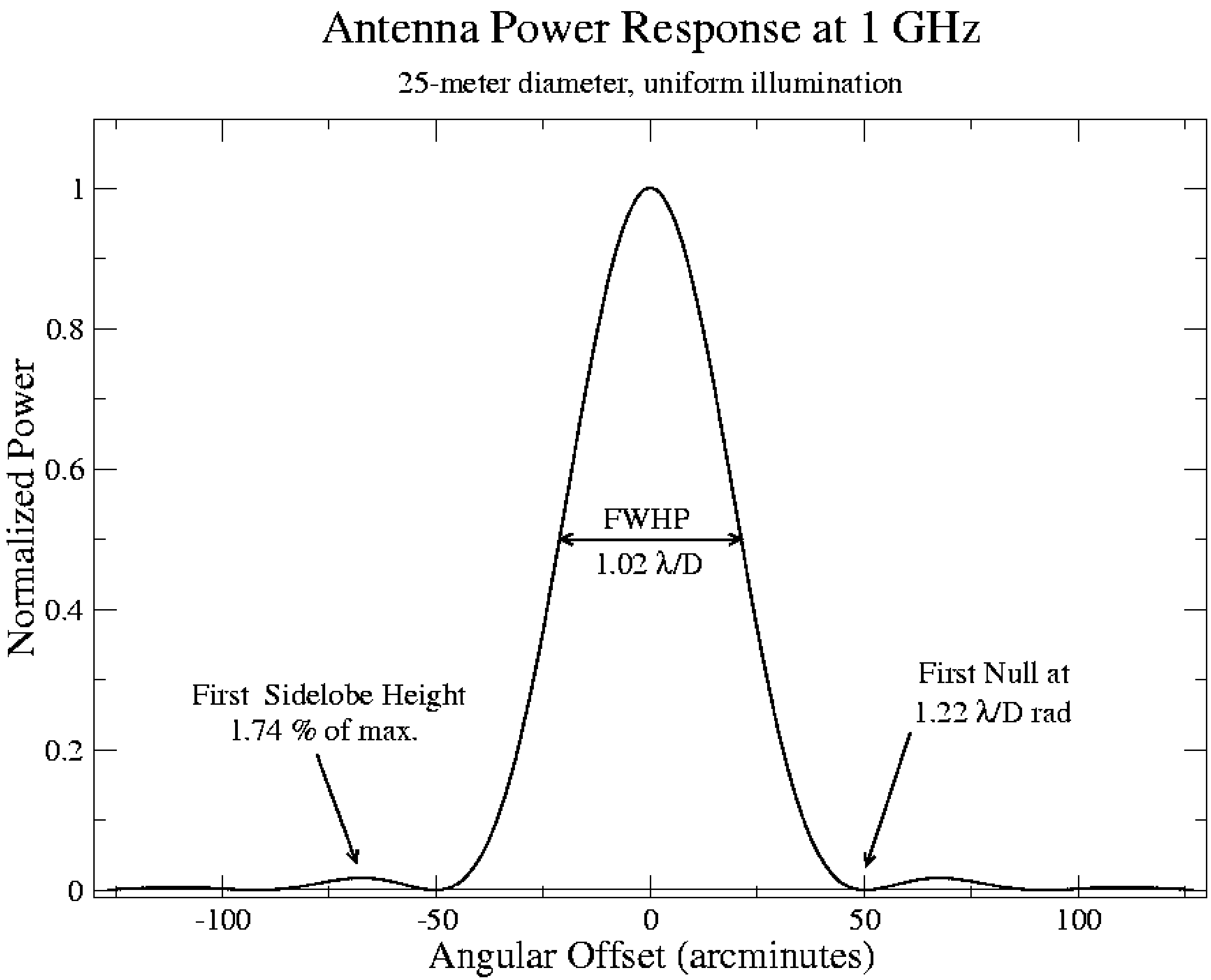
The Parabolic Reflector

- Key Point: Distance from incoming phase front to focal point is the same for all rays.
- The E-fields will thus all be in phase at the focus – the place for the receiver.



The Standard Parabolic Antenna Response

- The power response of a uniformly illuminated circular parabolic antenna of 25-meter diameter, at a frequency of 1 GHz.
- The FWHM, and the angle to the first null are both approximately λ/D (radians).
- There must always be some response outside the main beam – these are called the 'sidelobes'.



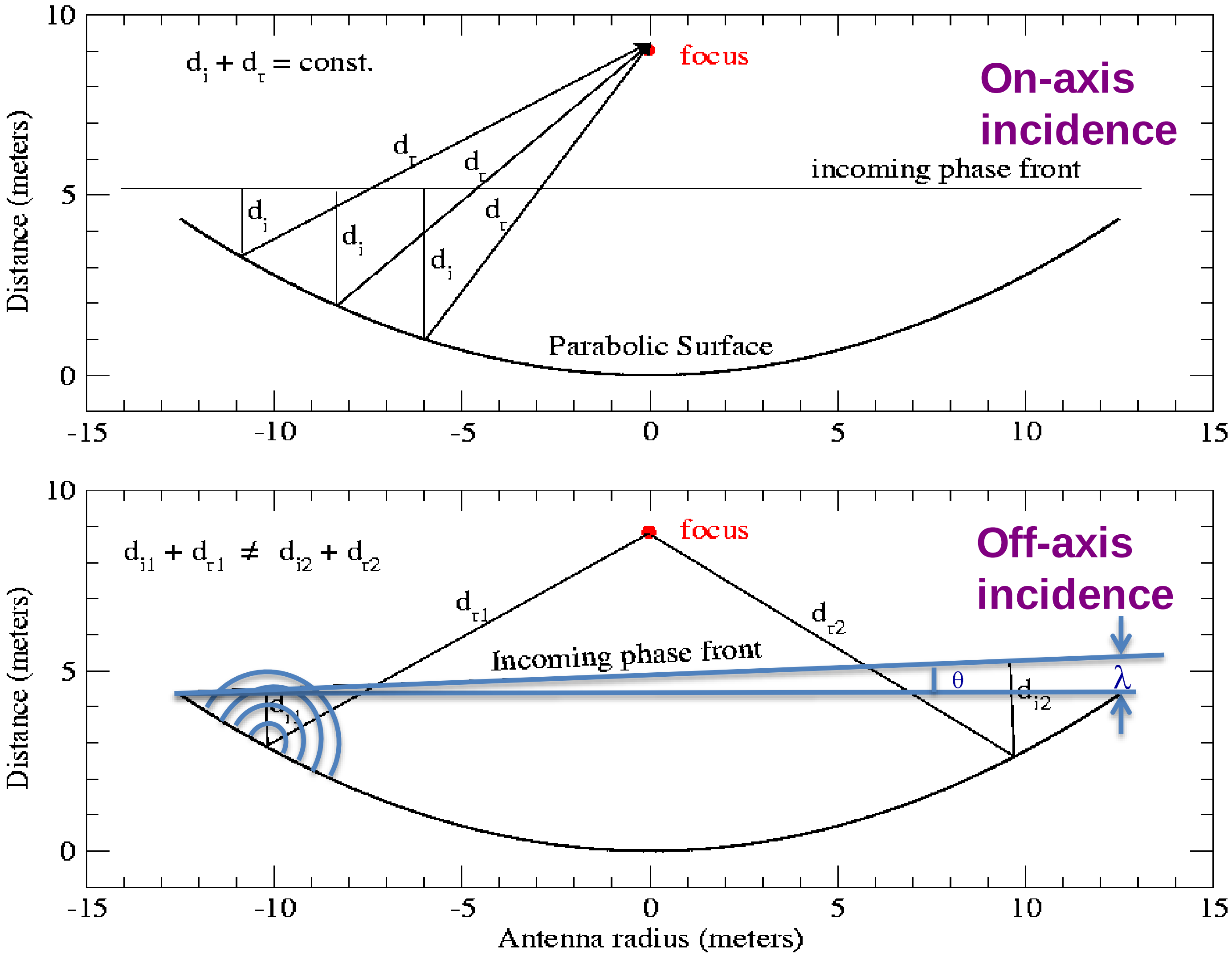
Beam Pattern Origin – Why is there a Null?

- An antenna's response is a result of coherent phase summation of the electric field at the focus.

- First null will occur at the angle where one extra wavelength of path is added across the full width of the aperture:

$\theta \sim \lambda/D$

(Why?)



Specifics: First Null, and First Sidelobe

- When the phase differential across the aperture is 1, 2, 3, … wavelengths, we get a null in the total received power.
  - The nulls appear at (approximately): $\theta = \lambda/D, 2\lambda/D, 3\lambda/D, \ldots$ radians.
- When the phase differential across the aperture is ~1.5, 2.5, 3.5, … wavelengths, we get a maximum in total received power.
  - But, each successive maximum is weaker than the last.
  - These maxima appear at (approximately): $\theta = 3\lambda/2D, 5\lambda/2D, 7\lambda/2D, \ldots$ radians.

Why Interferometry?

- Radio telescopes coherently sum electric fields over an aperture of size D.
- For this, diffraction theory applies – the angular resolution is:

$$\theta_{rad} \approx \lambda / D$$

- In 'practical' units:

$$\theta_{\mathrm{arcsec}} \approx 2\lambda_{\mathrm{cm}} / D_{\mathrm{km}}$$

- To obtain 1 arcsecond resolution at a wavelength of 21 cm, we require an aperture of ~42 km!
- The (currently) largest single, fully-steerable aperture is the 100-m antennas in Bonn, and Green Bank. Nowhere big enough!
- Can we synthesize an aperture of that size with pairs of antennas?
- The methodology of synthesizing a continuous aperture through summations of separated pairs of antennas is called 'aperture synthesis'.

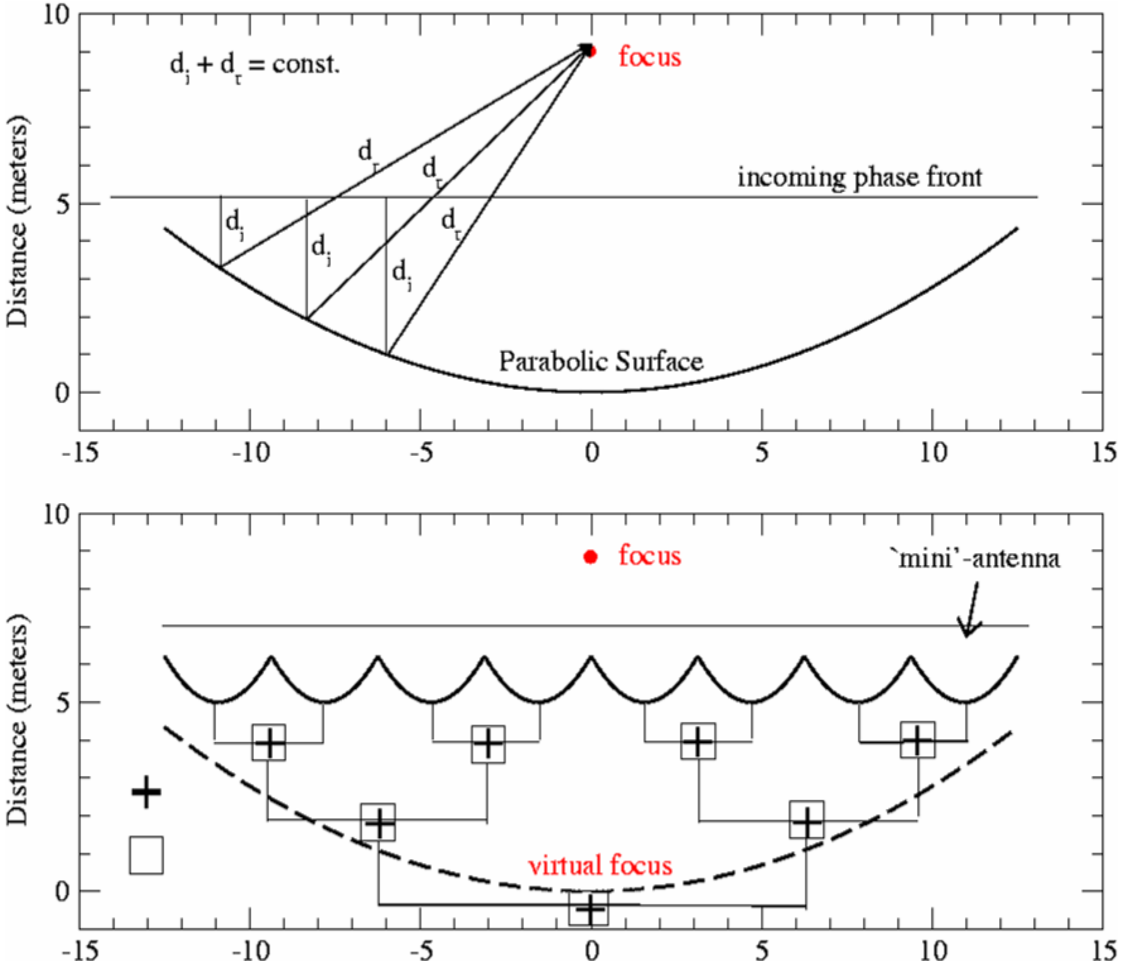
Interferometry – Basic Concept

- We don't need a single parabolic structure!

- We can consider a series of small antennas, whose individual signals are summed in a network.

- This is the basic concept of interferometry.

- Aperture Synthesis is an extension of this concept.



Quasi-Monochromatic Radiation

- Analysis is simplest if the fields are perfectly monochromatic.
- This is not possible – a perfectly monochromatic electric field would both have no power ($\Delta\nu = 0$), and would last forever!
- So we consider instead 'quasi-monochromatic' radiation, where the bandwidth $d\nu$ is finite, but very small. Then, for times periods $dt < 1/d\nu$, we can consider the field to have a constant amplitude and phase:

$$E_\nu(t) = E\cos(2\pi\nu t + \phi)$$

- Consider then the electric fields from a small sold angle $d\Omega$ about some direction **s**, within some small bandwidth $d\nu$, at frequency ν.
- The amplitude and phase remain unchanged to a time duration of order $dt \sim 1/d\nu$, after which new values of **E** and ϕ are needed.

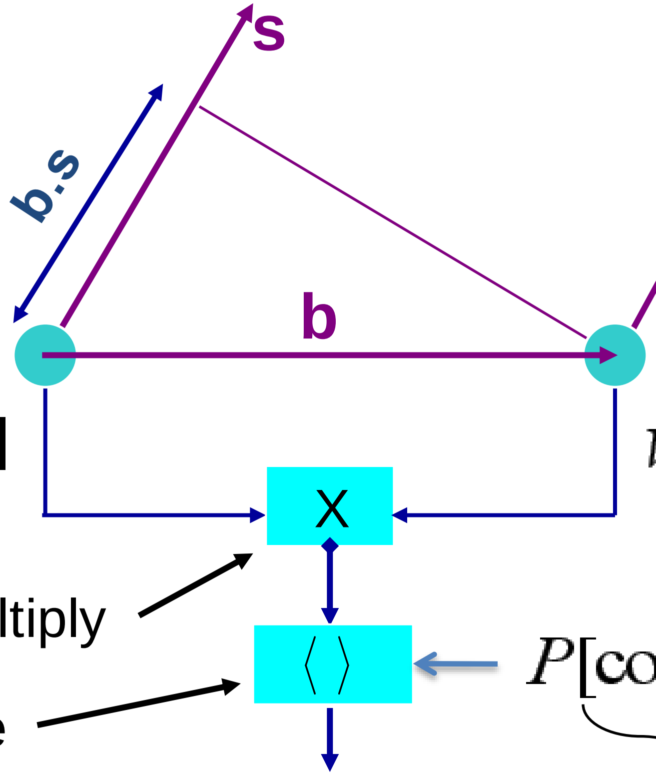
Simplifying Assumptions

- We now consider the most basic interferometer, and seek a relation between the characteristics of the product of the voltages from two separated antennas and the distribution of the brightness of the originating source emission.
- To establish the basic relations, the following simplifications are introduced:
  - Fixed in space – no rotation or motion
  - Quasi-monochromatic
  - No frequency conversions (an 'RF interferometer')
  - Single polarization
  - No propagation distortions (no ionosphere, atmosphere …)
  - Idealized electronics (perfectly linear, perfectly uniform in frequency and direction, perfectly identical for both elements, no added noise, …)

The Stationary, Quasi-Monochromatic Radio-Frequency Interferometer

Geometric Time Delay

$$\tau_g = \mathbf{b} \cdot \mathbf{s} / c$$



$$V_1 = E\cos[\omega(t - \tau_g)]$$

$$V_2 = E\cos(\omega t)$$

The path lengths from sensors to multiplier are assumed equal!

multiply

average

$$P[\underbrace{\cos(\omega\tau_g)}_{\text{Unchanging}} + \underbrace{\cos(2\omega t - \omega\tau_g)}_{\text{Rapidly varying,}}]$$

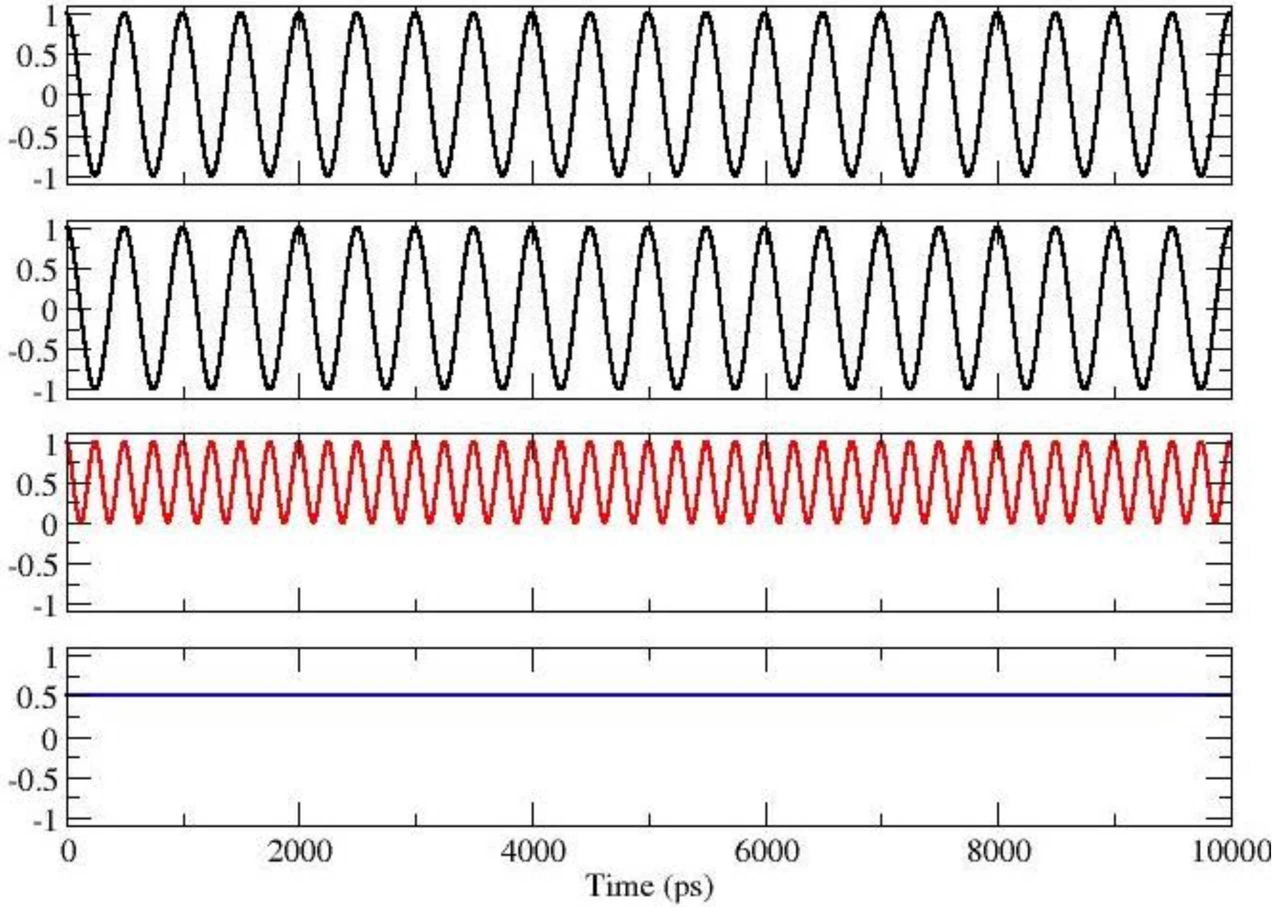
$$R_C = P\cos(\omega\tau_g)$$

Pictorial Example: Signals In Phase

2 GHz Frequency, with voltages in phase:

$\mathbf{b.s} = n\lambda$, or $\tau_g = n/\nu$

- Antenna 1 Voltage
- Antenna 2 Voltage
- Product Voltage
- Average

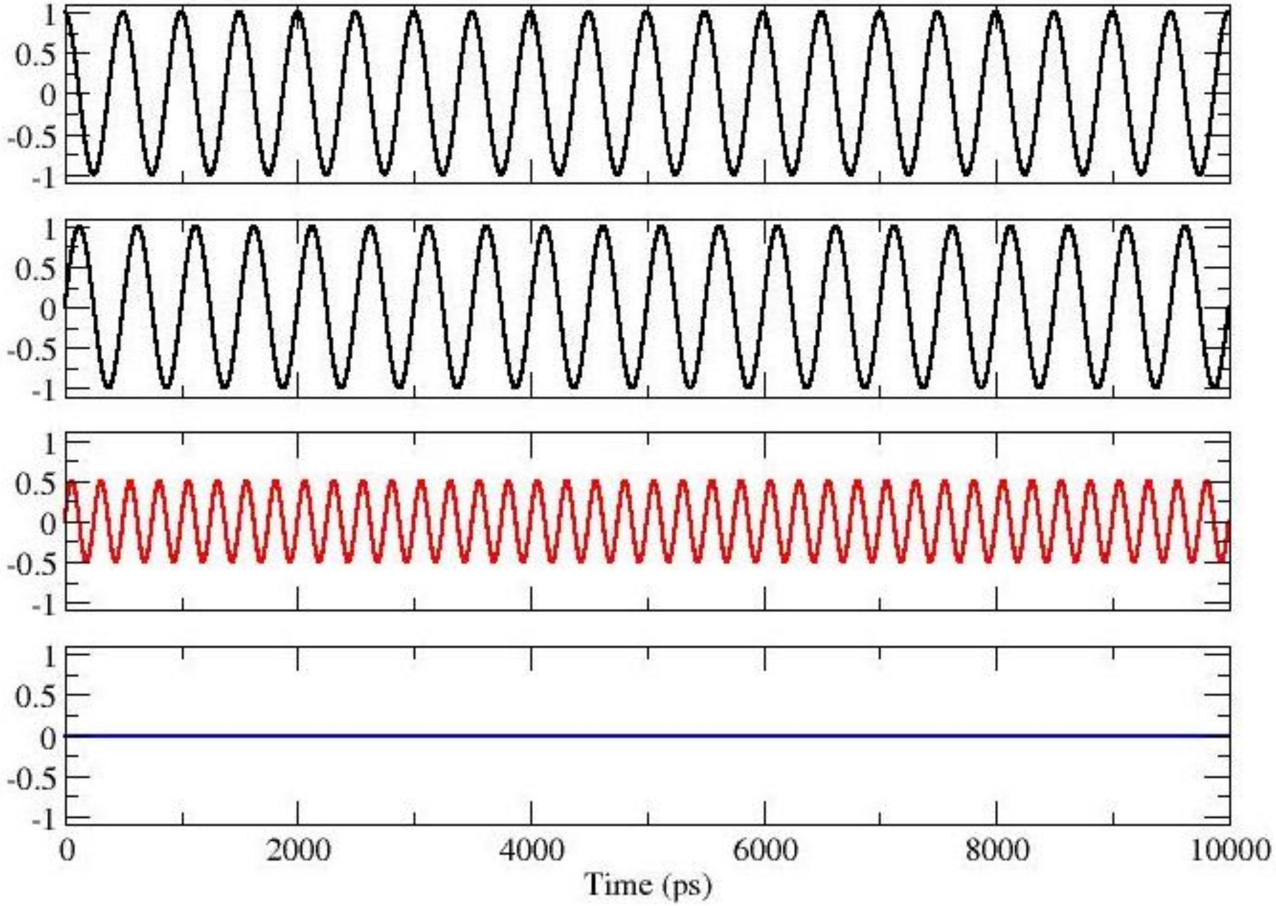


Pictorial Example: Signals in Quad Phase

2 GHz Frequency, with voltages in quadrature phase:

$b.s=(n \pm \frac{1}{4})\lambda, \ \tau_g = (4n \pm 1)/4\nu$

- Antenna 1 Voltage
- Antenna 2 Voltage
- Product Voltage
- Average

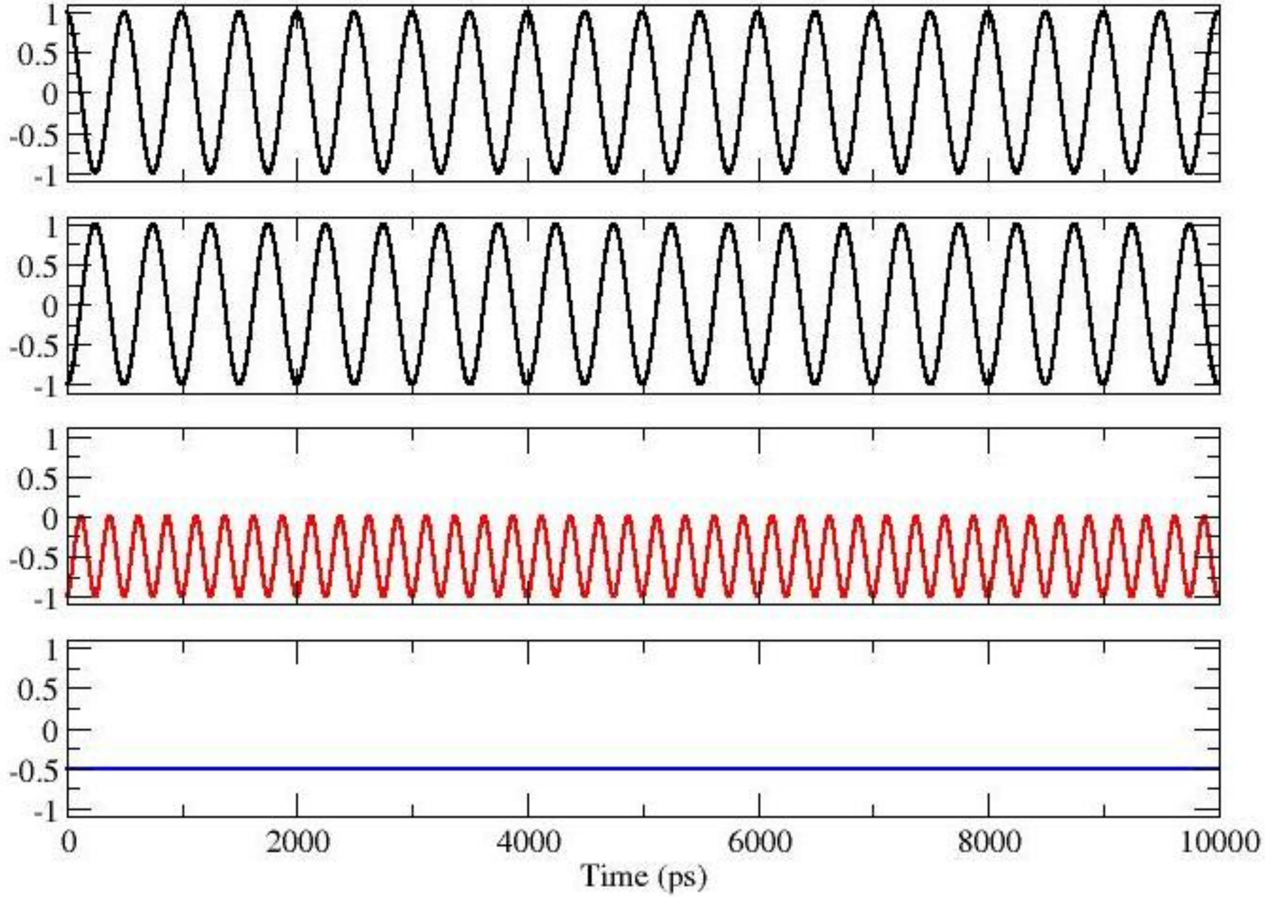


Pictorial Example: Signals out of Phase

2 GHz Frequency, with voltages out of phase:

$b.s=(n \pm \frac{1}{2})\lambda \qquad \tau_g = (2n \pm 1)/2\nu$

- Antenna 1 Voltage
- Antenna 2 Voltage
- Product Voltage
- Average



Some General Comments

- The averaged product R_C is dependent on the received power, $P = E^2/2$ and geometric delay, τ_g, and hence on the baseline orientation and source direction:

$$R_C = P\cos(\omega\tau_g) = P\cos\left(2\pi\frac{\mathbf{b}\cdot\mathbf{s}}{\lambda}\right)$$

- Note that R_C is not a a function of:
  - The time of the observation -- provided the source itself is not variable!
  - The location of the baseline -- provided the emission is in the far-field.
  - The actual phase of the incoming signal – the distance of the source does not matter, provided it is in the far-field.

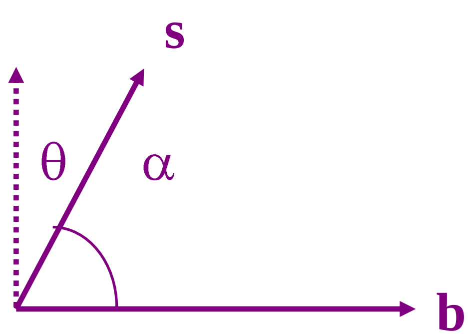
- The strength of the product is dependent on the antenna areas and electronic gains – but these factors can be calibrated for.

Pictorial Illustrations

- To illustrate the response, expand the dot product in one dimension:

$$\frac{\mathbf{b}\bullet\mathbf{s}}{\lambda} = u\cos\alpha = u\sin\theta = ul$$

- Here, **u = b/λ is the baseline length in wavelengths**, and θ is the angle w.r.t. the plane perpendicular to the baseline.
- $l = \cos\alpha = \sin\theta$ is the direction cosine



- Consider the response R_c, as a function of angle, for two different baselines with u = 10, and u = 25 wavelengths:

$$R_C = \cos(2\pi ul)$$

Whole-Sky Response

- Top:

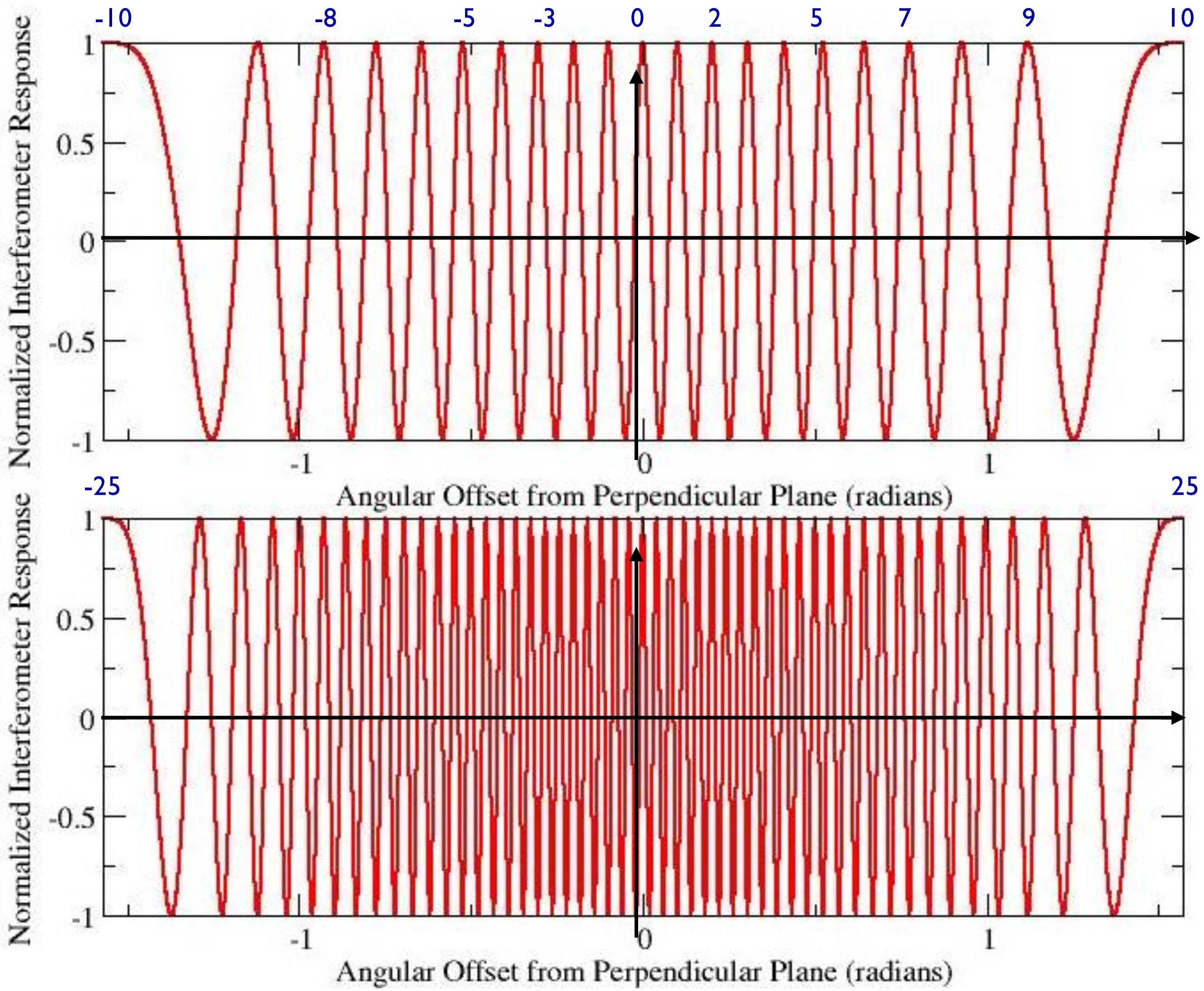
  u = 10

There are 20 whole fringes over the hemisphere.

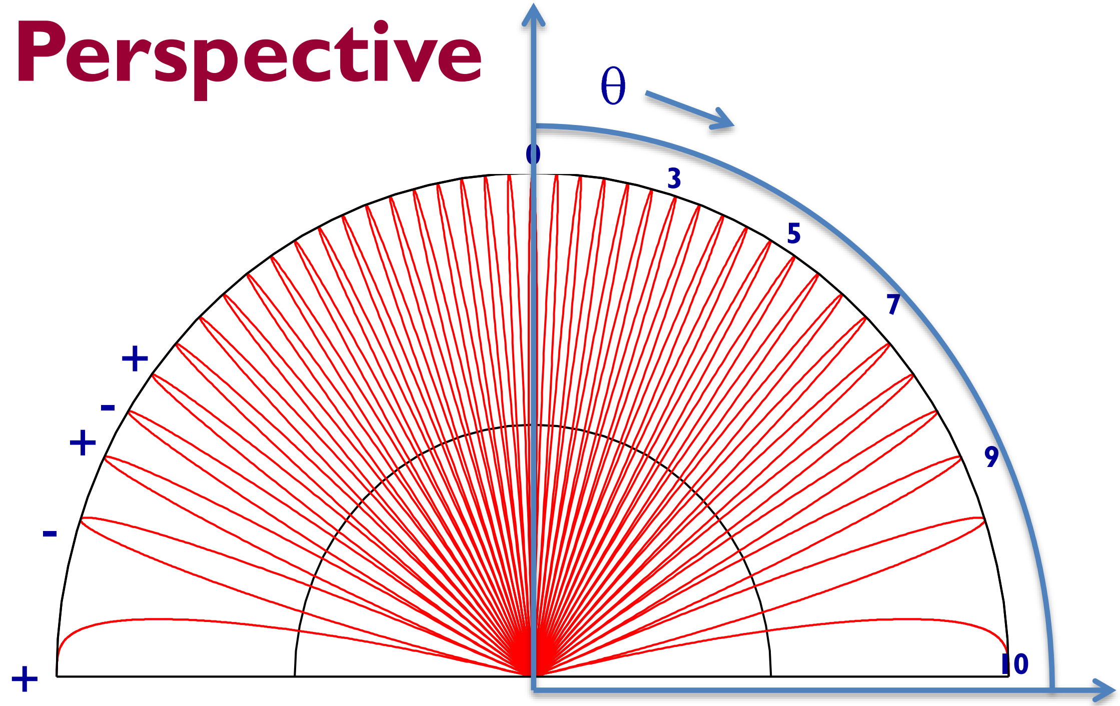
- Bottom:

  u = 25

There are 50 whole fringes over the hemisphere



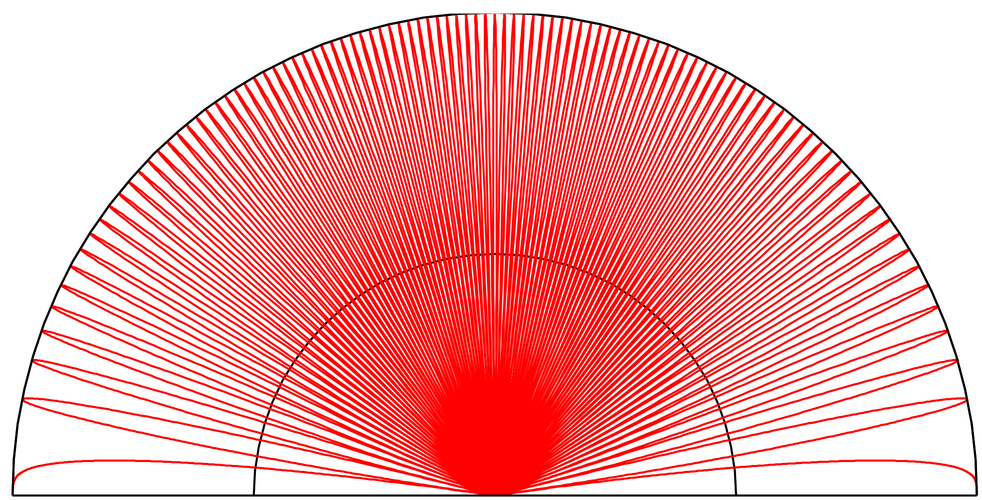
From an Angular Perspective



Top Panel:

The absolute value of the response for u = 10, as a function of angle.

The 'lobes' of the response pattern alternate in sign.



Bottom Panel:

The same, but for u = 25.

Angular separation between lobes (of the same sign) is

$\delta\theta \sim 1/u = \lambda/b$ radians.

Hemispheric Pattern

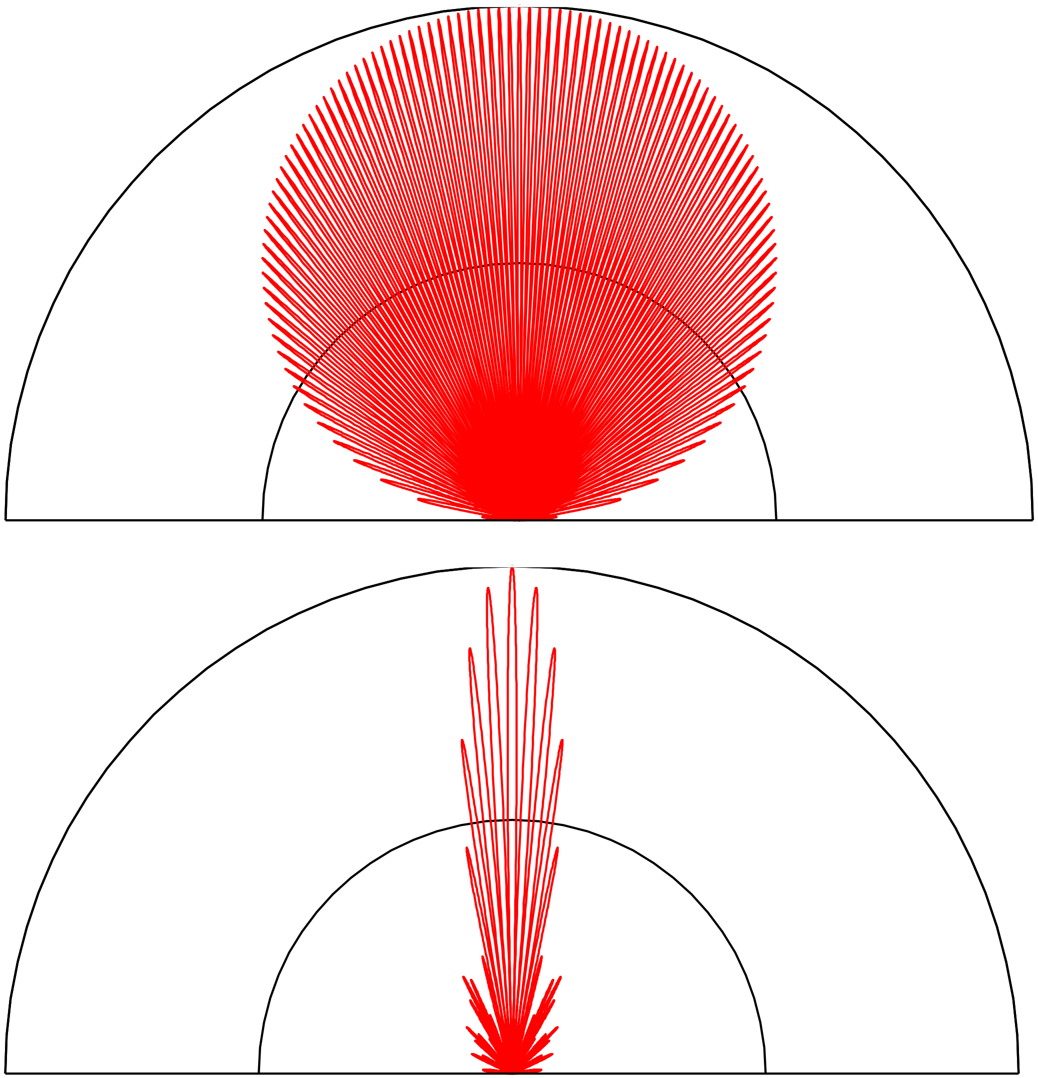
- The preceding plot is a meridional cut through the hemisphere, oriented along the baseline vector.
- In two-dimensional space, the coherence pattern consists of a series of coaxial cones, oriented along the baseline vector.
- The figure is a two-dimensional representation when u = 4.
- As viewed along the baseline vector, the fringes show a 'bulls-eye' pattern – concentric circles.

The Effect of the Sensor

- The patterns shown presume the sensor has isotropic response.
- This is a convenient assumption, but (sadly, for some) doesn't represent reality.
- Real sensors impose their own patterns, which modulate the amplitude and phase of the output.
- Large sensors (a.k.a. 'antennas') have very high directivity -- very useful for some applications.

The Effect of Sensor Patterns

- Sensors (or antennas) are not isotropic, and have their own responses.
- **Top Panel:** The interferometer pattern with a cos(θ)-like sensor response.
- **Bottom Panel:** A multiple-wavelength aperture antenna has a narrow beam, but also sidelobes.



The Response from an Extended Source

- The response from an extended source is obtained by summing the responses at each antenna to all the emission over the sky, multiplying the two, and averaging:

$$R_C = \left\langle \int V_1 d\Omega_1 \int V_2 d\Omega_2 \right\rangle$$

- The averaging and integrals can be interchanged and, **providing the emission is spatially incoherent**, we get

$$R_C = \iint I_\nu(\mathbf{s}) \cos(2\pi\nu \mathbf{b}\cdot\mathbf{s}/c) d\Omega$$

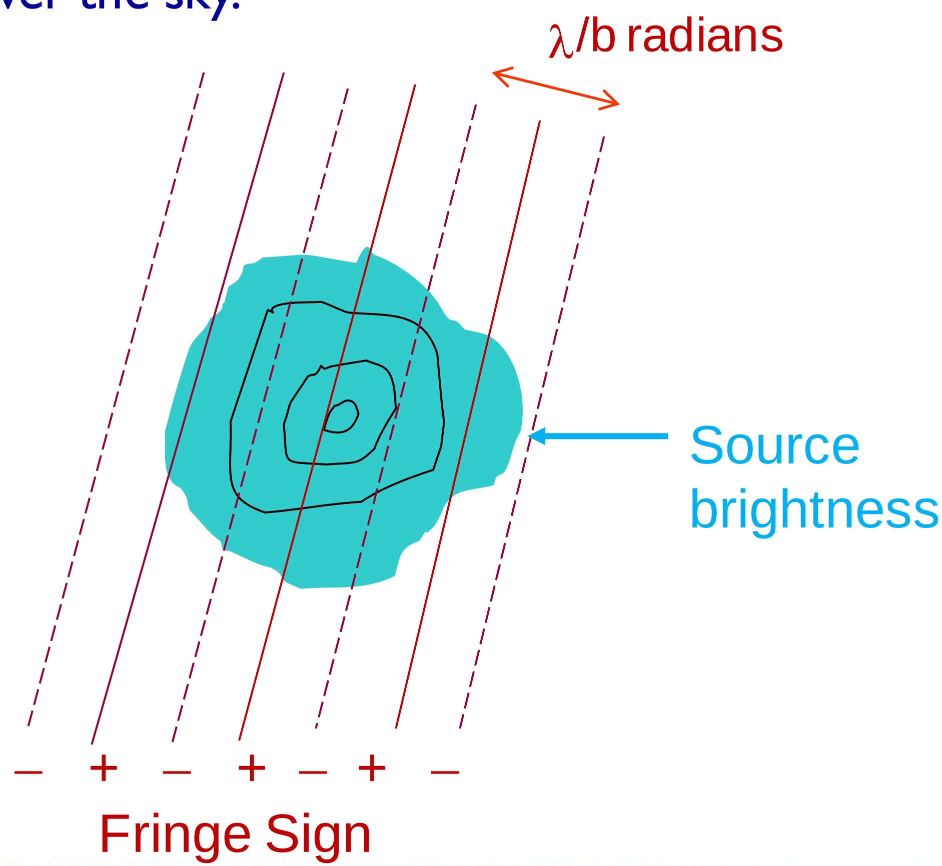
- This expression links what we want – the source brightness on the sky, $I_\nu(\mathbf{s})$, – to something we can measure - R_C, the interferometer response.
- Can we recover $I_\nu(\mathbf{s})$ from observations of R_C?

A Schematic Illustration in 2-D

- The correlator can be thought of 'casting' a cosinusoidal coherence pattern, of angular scale ~λ/b radians, onto the sky.

- The correlator multiplies the source brightness by this coherence pattern, and integrates (sums) the result over the sky.

- Orientation set by baseline geometry.
- Fringe separation set by (projected) baseline length and wavelength.
  - Long baseline gives close-packed fringes
  - Short baseline gives widely-separated fringes
- Physical location of baseline unimportant, provided source is in the far field.



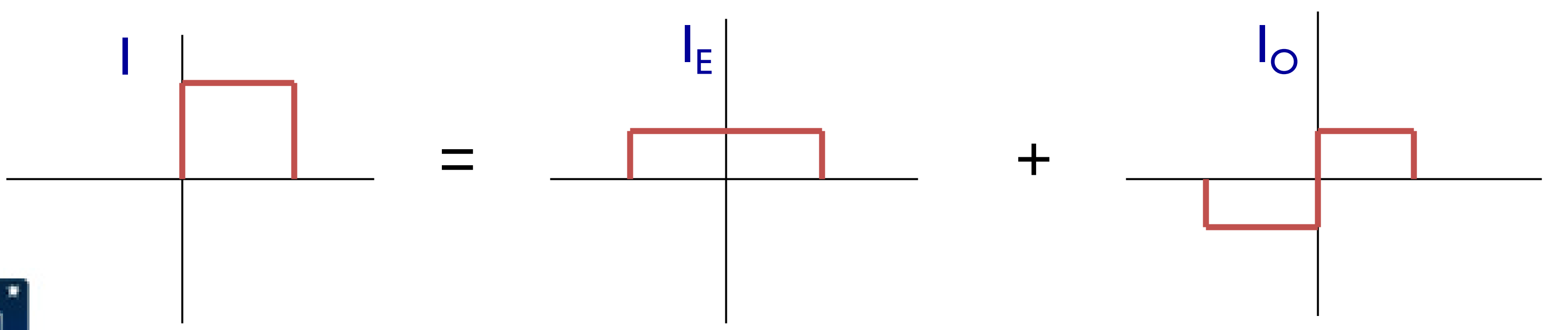
Odd and Even Functions

- Any real function, I(x,y), can be expressed as the sum of two real functions which have specific symmetries:

$$I(x,y) = I_E(x,y) + I_O(x,y)$$

An even part: $$I_E(x,y) = \frac{I(x,y) + I(-x,-y)}{2} = I_E(-x,-y)$$

An odd part: $$I_O(x,y) = \frac{I(x,y) - I(-x,-y)}{2} = -I_O(-x,-y)$$



But One Correlator is Not Enough!

- The correlator response, R_c:

$$R_C = \iint I_\nu(\mathbf{s}) \cos(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega$$

  is only sensitive to the even part of the sky brightness.

- Suppose that the source of emission has a component with odd symmetry:

$$I_O(s) = -I_O(-s)$$

- Since the cosine fringe pattern is even, the response of our interferometer to the odd brightness distribution is 0!

$$R_C = \iint I_O(\mathbf{s}) \cos(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega = 0$$

- Hence, we need more information if we are to completely recover the source brightness.

Why Two Correlations are Needed

- The integration of the cosine response, R_c, over the source brightness is sensitive to only the even part of the brightness:

$$R_C = \iint I(\mathbf{s})\cos(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)d\Omega = \iint I_E(\mathbf{s})\cos(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)d\Omega$$

  since the integral of an odd function (I_O) with an even function (cos x) is zero.
- To recover the 'odd' part of the intensity, I_O, we need an 'odd' fringe pattern. Let us replace the 'cos' with 'sin' in the integral.
- We get:

$$R_S = \iint I(\mathbf{s})\sin(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)d\Omega = \iint I_O(\mathbf{s})\sin(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)d\Omega$$

  since the integral of an even times an odd function is zero.
- Thus, the even part of the image is 'seen' by the COS correlation, while the odd part is 'seen' by the SIN correlation.
- Hence, to get the full image, we need both.

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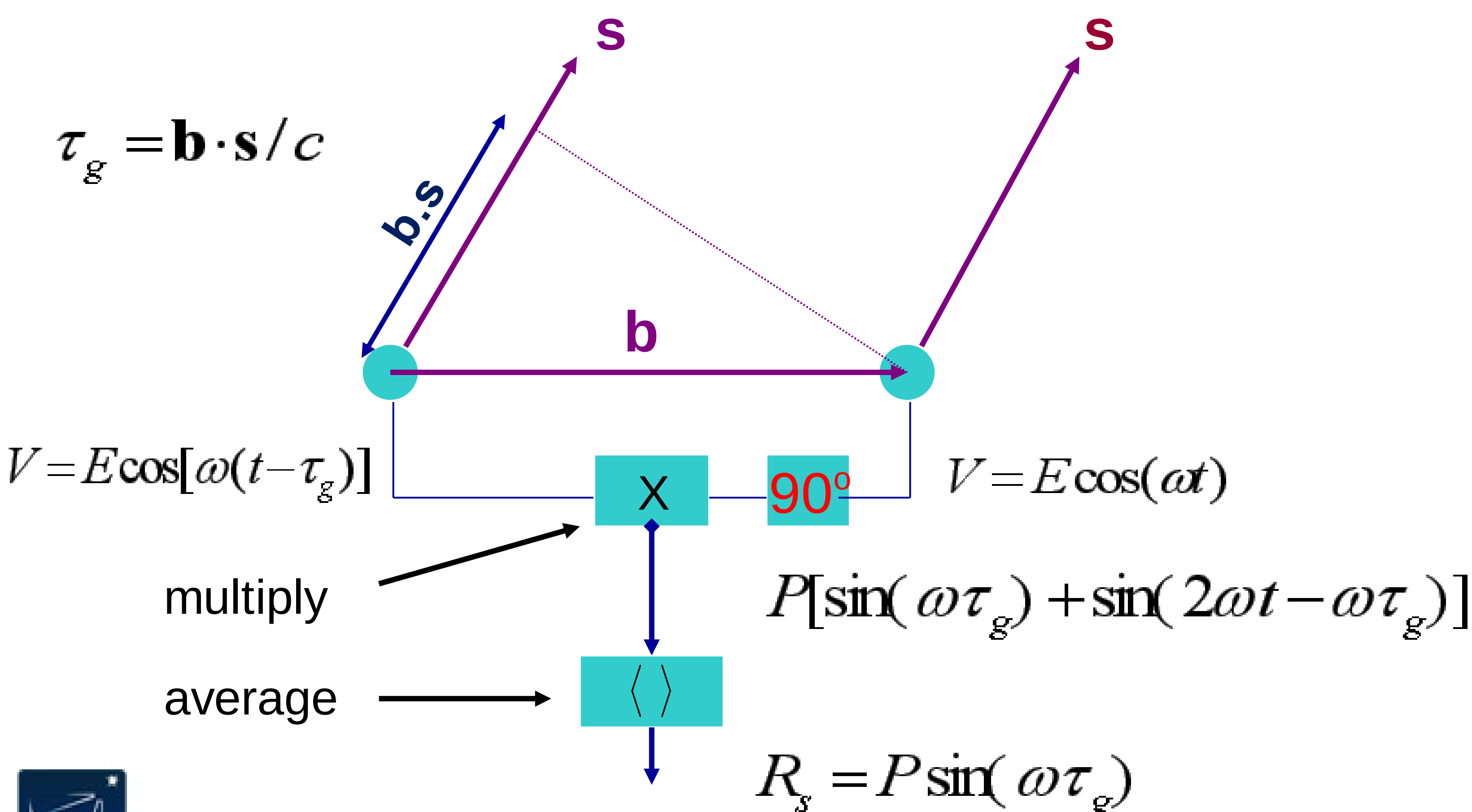
Making a SIN Correlator

- We generate the 'sine' pattern by inserting a 90 degree phase shift in one of the signal paths.



Define the Complex Visibility

- We now DEFINE a complex function, the complex visibility, V, from the two independent (real) correlator outputs R_C and R_S:

$$V = R_C - iR_S = Ae^{-i\phi}$$

where

$$A = \sqrt{R_C^2 + R_S^2}$$

$$\phi = \tan^{-1}\left(\frac{R_S}{R_C}\right)$$

- This gives us a beautiful and useful relationship between the source brightness, and the response of an interferometer:

$$V_\upsilon(\mathbf{b}) = R_C - iR_S = \iint I_\nu(s) e^{-2\pi i \nu \mathbf{b}\cdot\mathbf{s}/c} d\Omega$$

- Under some circumstances, this is a 2-D Fourier transform, giving us a well established way to recover $I(\mathbf{s})$ from $V(\mathbf{b})$.

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The Complex Correlator and Complex Notation

- A correlator which produces both 'Real' and 'Imaginary' parts – or the Cosine and Sine fringes, is called a 'Complex Correlator'
  - For a complex correlator, think of two independent sets of projected sinusoids, 90 degrees apart on the sky.
  - In our scenario, both components are necessary, because we have assumed there is no motion – the 'fringes' are fixed on the source emission, which is itself stationary.
- The complex output of the complex correlator also means we can use complex analysis throughout: Let:

$$V_1 = A\cos(\omega t) = \mathrm{Re}\left(Ae^{-i\omega t}\right)$$

$$V_2 = A\cos[\omega(t - \mathbf{b}\cdot\mathbf{s}/c)] = \mathrm{Re}\left(Ae^{-i\omega(t-\mathbf{b}\bullet\mathbf{s}/c)}\right)$$

- Then:

$$P_{corr} = \left\langle V_1 V_2^{*} \right\rangle = A^2 e^{-i\omega\,\mathbf{b}\bullet\mathbf{s}/c}$$

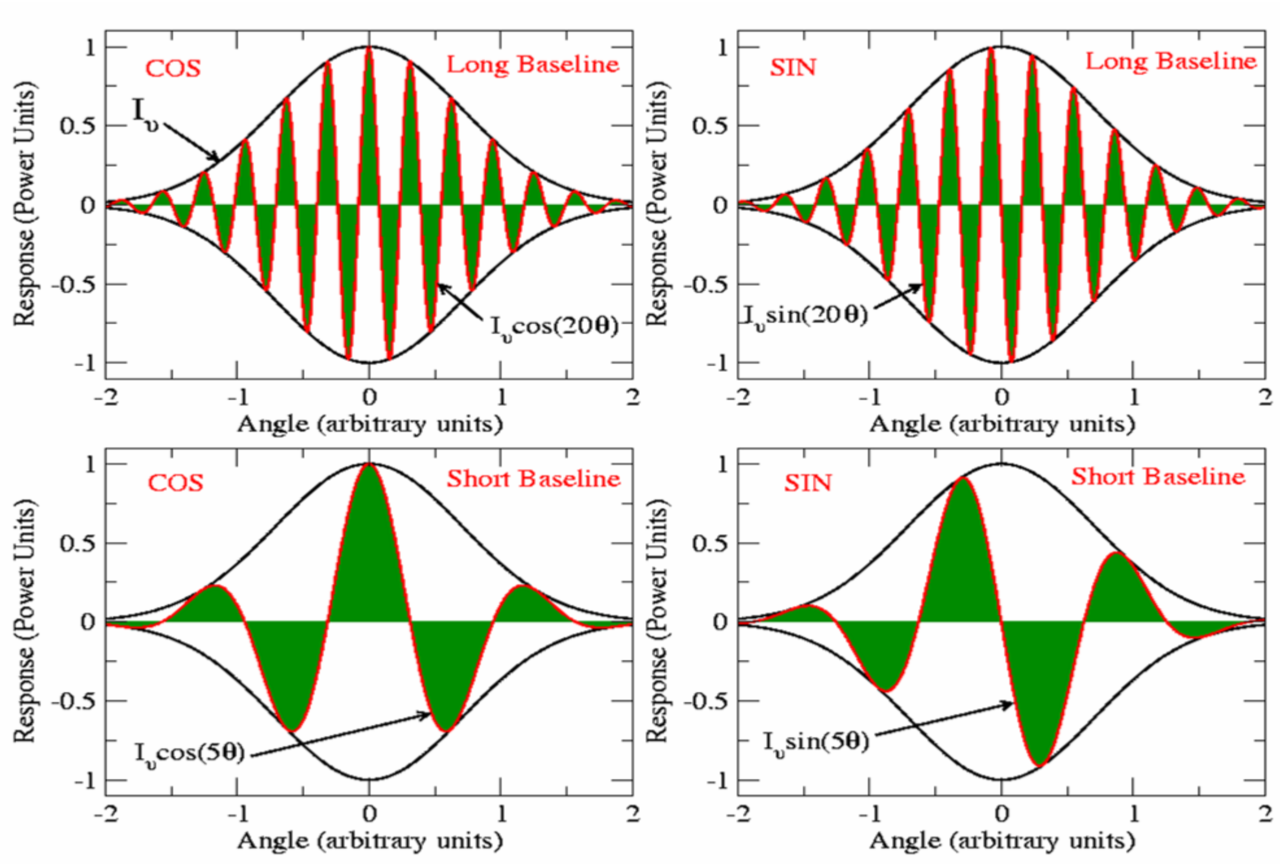
Picturing the Visibility

- The source brightness is Gaussian, shown in black.
- The interferometer 'fringes' are in red.
- The visibility is the integral of the product – the net dark green area.

R_C R_S

Long Baseline

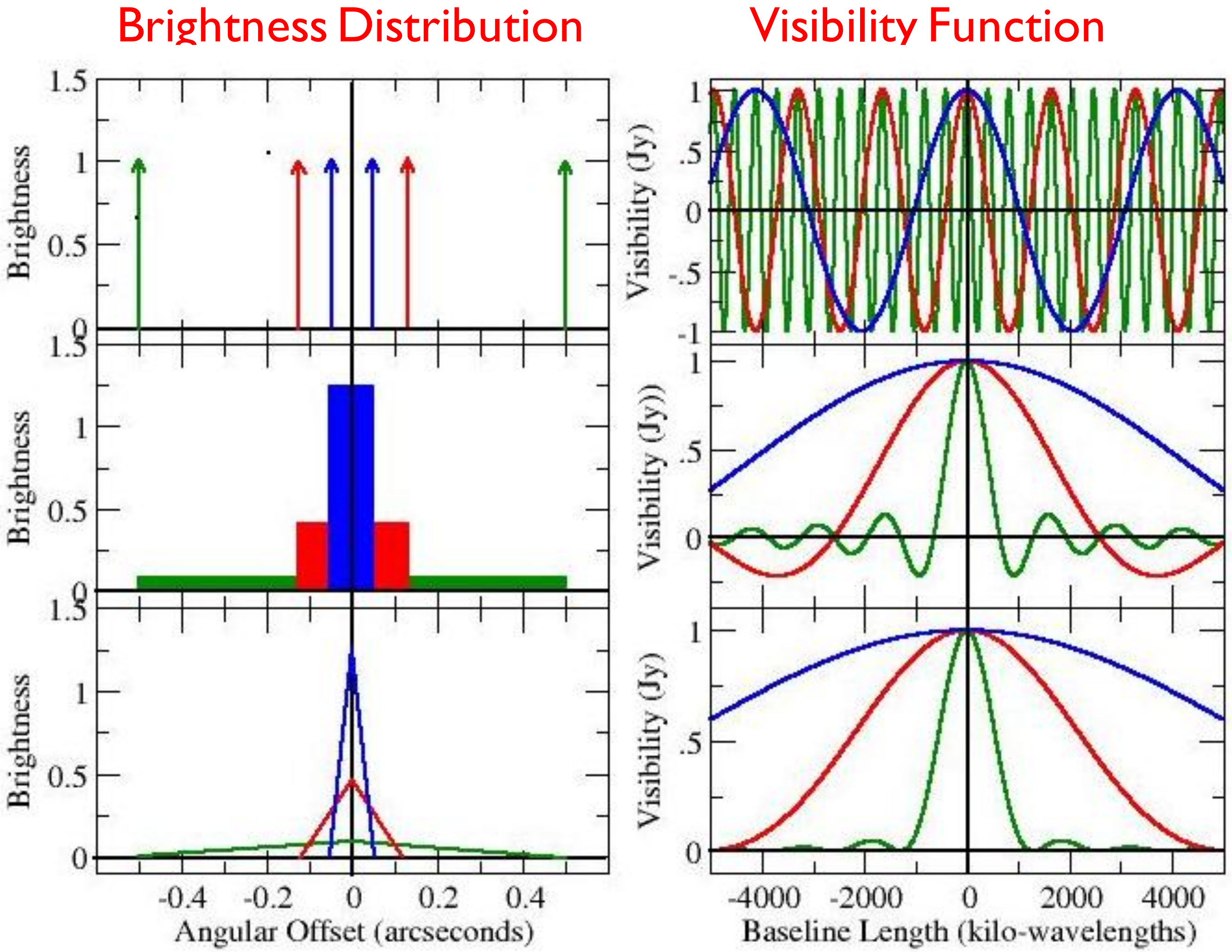
Short Baseline



Examples of 1-Dimensional Visibilities

- Simple pictures are easy to make illustrating 1-dimensional visibilities.

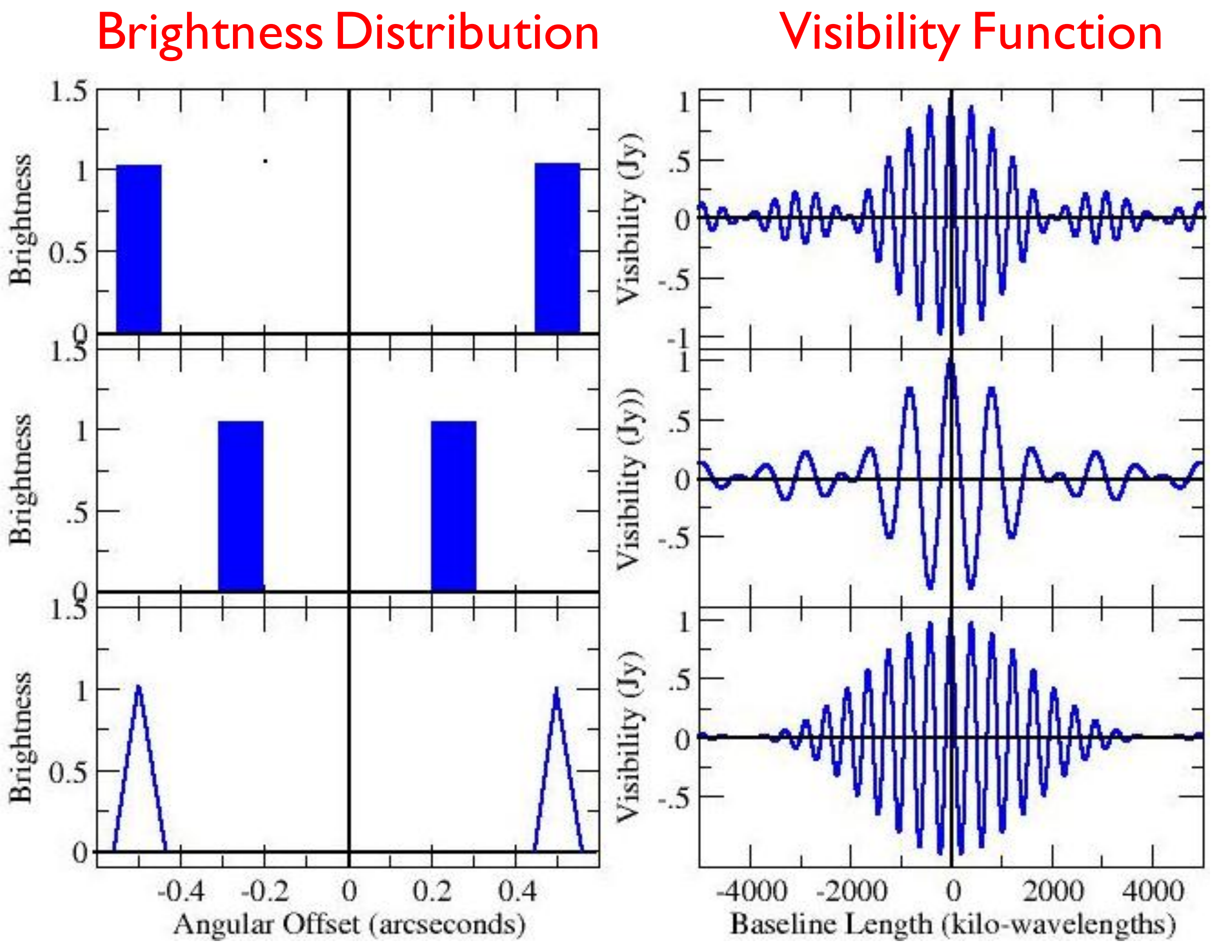
- Unresolved Doubles
- Uniform
- Central Peaked



More Examples

- Simple pictures are easy to make illustrating 1-dimensional visibilities.

- Resolved Double
- Resolved Double
- Central Peaked Double



Basic Characteristics of the Visibility

- For a zero-spacing interferometer, we get the 'single-dish' (total-power) response.
- As the baseline gets longer, the visibility amplitude will in general decline.
- When the visibility is close to zero, the source is said to be 'resolved out'.
- Interchanging antennas in a baseline causes the phase to be negated – the visibility of the 'reversed baseline' is the complex conjugate of the original.
- Mathematically, the visibility is Hermitian, because the brightness is a real function.

Some Comments on Visibilities

- The Visibility is a unique function of the source brightness.
- The two functions are related through a Fourier transform. $V_\nu(u,v) \Leftrightarrow I(l,m)$
- An interferometer, at any one time, makes one measure of the visibility, at baseline coordinate (u,v).
- Sufficient knowledge of the visibility function (as derived from an interferometer) will provide us a reasonable estimate of the source brightness.
- How many is 'sufficient', and how good is 'reasonable'?
- These simple questions do not have easy answers…