Noise and Interferometry

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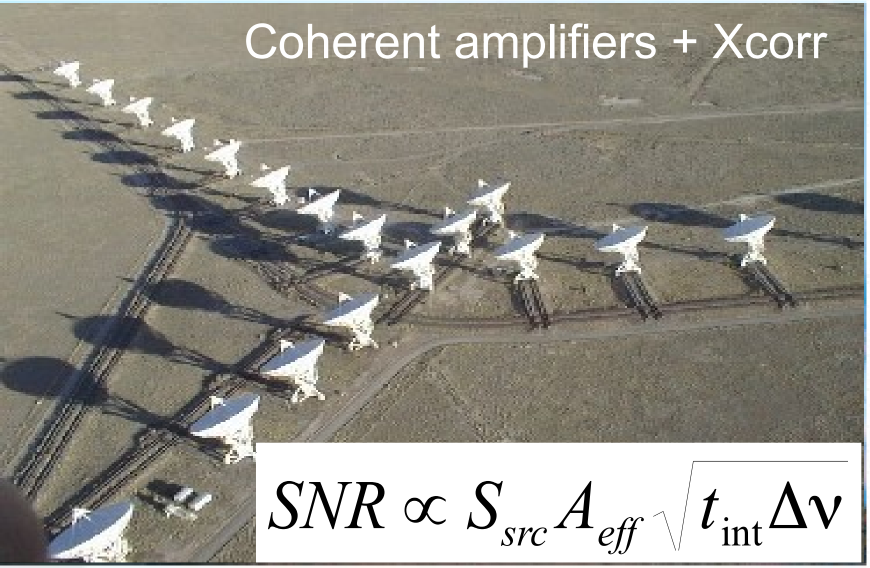

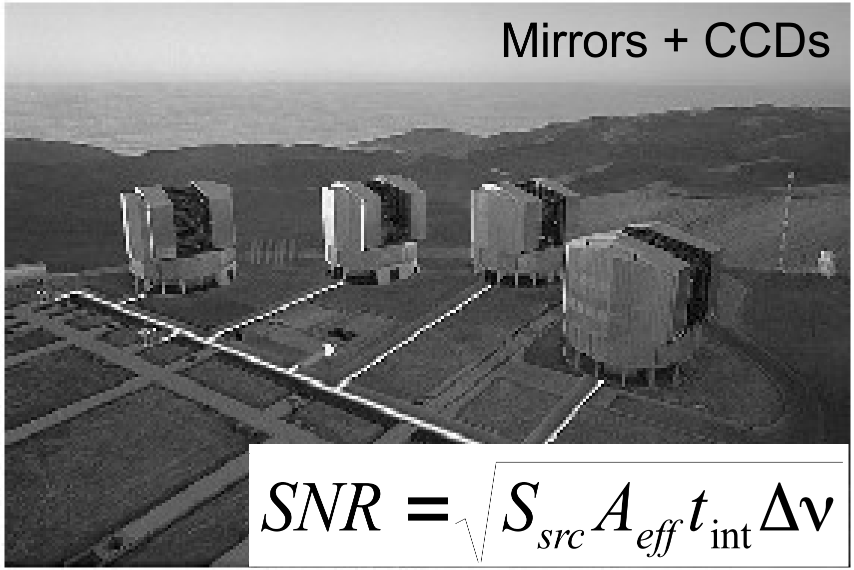

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References

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- ‘Thermal noise and correlations in photon detection,’ J. Zmuidzinas 2000, Applied Optics, 42, 4989
- ‘Bolometers for infrared and millimeter waves,’ P. Richards, J. 1994, Applied Physics, 76, 1
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- ‘Tools of Radio Astronomy,’ Rohlfs & Wilson (Springer)

Some definitions

I_ν (or B_ν) = Surface Brightness : erg/s/cm^2/Hz/sr
(= intensity)

S_ν = Flux density : erg/s/cm^2/Hz $\int I_\nu \Delta\Omega$

S = Flux : erg/s/cm^2 $\int I_\nu \Delta\Omega\Delta\nu$

P = Power received : erg/s $\int I_\nu \Delta\Omega\Delta\nu\Delta A_{tel}$

E = Energy : erg $\int I_\nu \Delta\Omega\Delta\nu\Delta A_{tel}\Delta t$

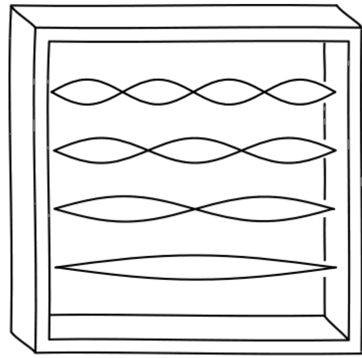
Radiometer Equation (interferometer)

$$S_{rms} = \frac{2kT_{sys}}{A_{eff}\sqrt{N_A(N_A - 1)t_{\mathrm{int}}\Delta\nu}}$$

Physically motivate terms

- Photon statistics: wave noise vs. shot noise (radio vs. optical)
- Quantum noise of coherent amplifiers
- Temperature in Radio Astronomy (Johnson-Nyquist resistor noise, Antenna Temp, Brightness Temp)
- Number of independent measurements of wave form (central limit theorem)
- Some interesting consequences

Concept 1: Photon statistics = Bose-Einstein statistics for gas without number conservation



Thermal equilibrium => **Planck distribution function** (Reif 9.5.4)

$$\langle n_s \rangle = (e^{h\nu_s/kT} - 1)^{-1}$$

n_s = photon occupation number, relative number in state s

= number of photons in standing-wave mode in box at temperature T

= number of photons/s/Hz in (diffraction limited) beam in free space[a] (Richards 1994, J.Appl.Phys)

Photon noise: variance in # photons arriving each second in free space beam

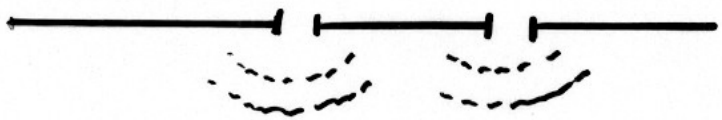
$$\langle \Delta n_s^2 \rangle \equiv \langle (n_s - \langle n_s \rangle)^2 \rangle = \langle n_s \rangle + \langle n_s \rangle^2 \qquad \text{(Reif 9.5.6)}$$

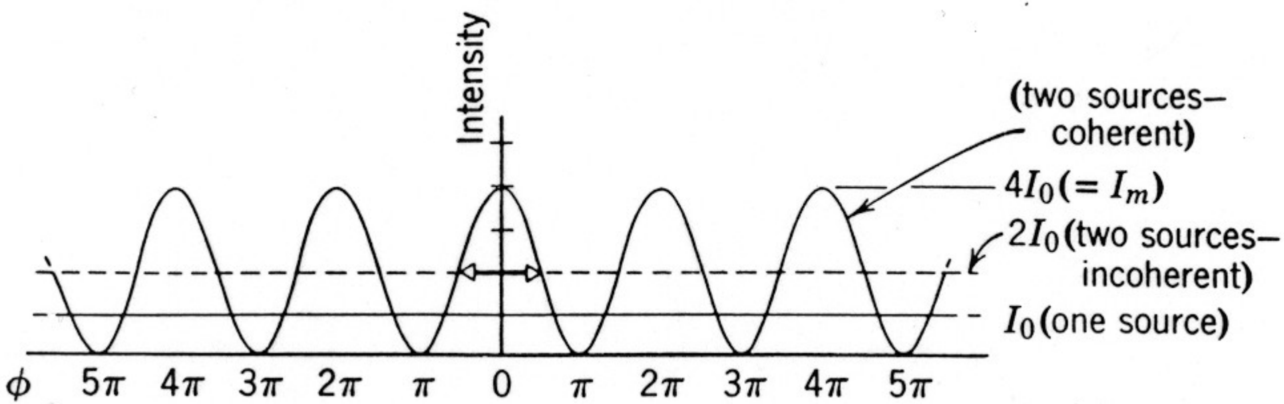
$$\langle n_s \rangle = \text{Poisson Stats}(n_s < 1) = \text{shot noise (noise} \propto \sqrt{n_s}) \qquad \text{Optical}$$

$$\langle n_s \rangle^2 = \text{Wave noise } (n_s > 1) \text{ (noise} \propto n_s) \qquad \text{Radio}$$

[a]Consider $T_B = T_{sys}$ for a diffraction limited beam

Physical motivation for wave noise: coherence



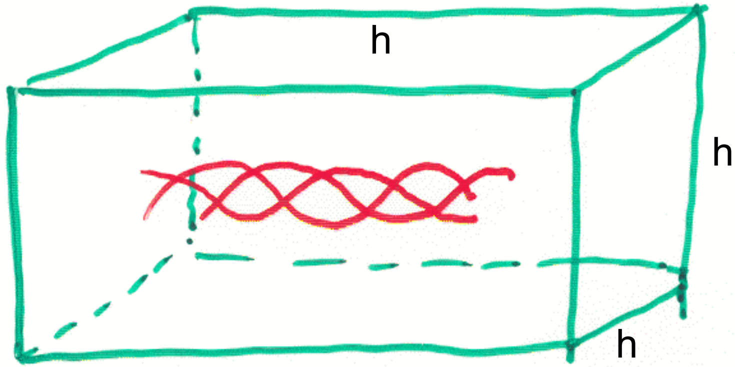


Single Source : $I \propto V^2 =$ '1 photon'

Two incoherent sources : $I \propto (V_1^2 + V_2^2) =$ '2 photons'

Two coherent sources : $I \propto (\vec{V}_1 + \vec{V}_2)^2 =$ '0 to 4 photons'

‘Bunching of Bosons’: photons can occupy exact same quantum state. Restricting phase space (ie. bandwidth and sampling time) leads to interference within the beam.

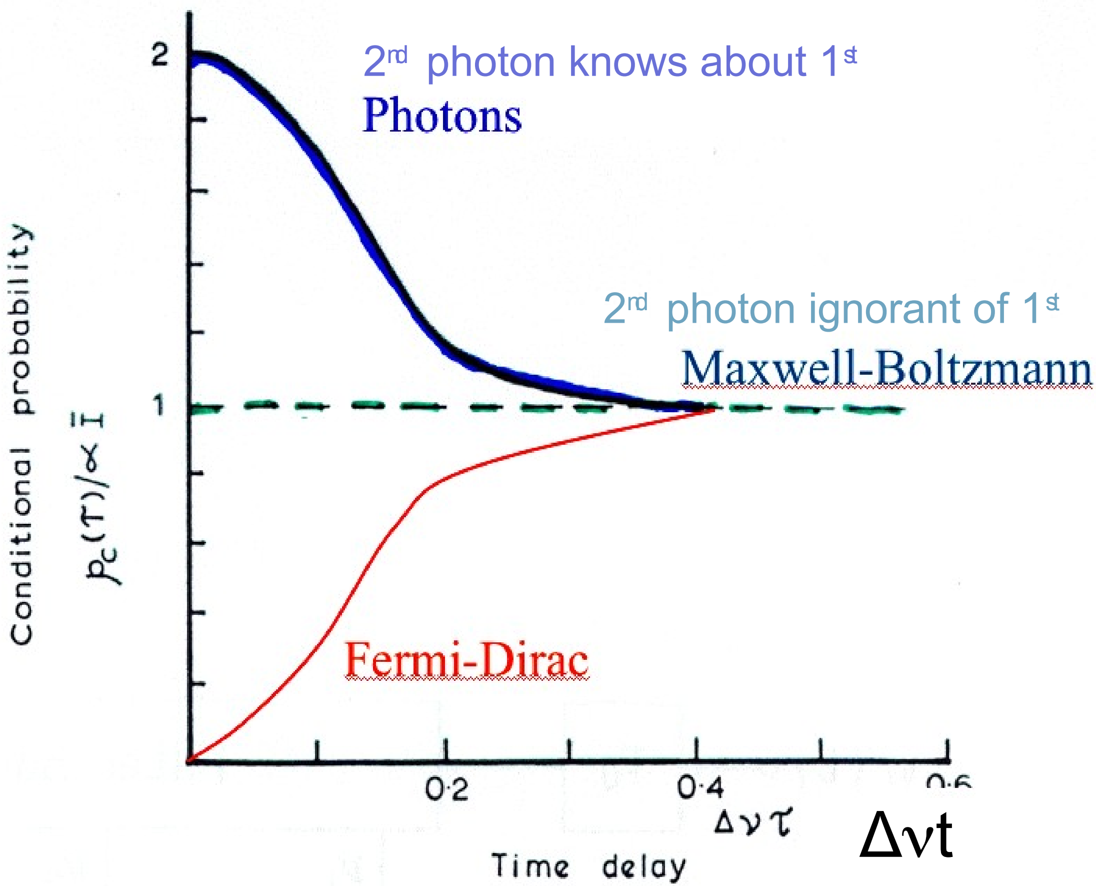


“Think then, of a stream of wave packets each about c/Δν long, in a random sequence. There is a certain probability that two such trains accidentally overlap. When this occurs they interfere and one may find four photons, or none, or something in between as a result. It is proper to speak of interference in this situation because the conditions of the experiment are just such as will ensure that these photons are in the same quantum state. **<u>To such interference one may ascribe the ‘abnormal’ density fluctuations in any assemblage of bosons</u>**.” **<u>(= wave noise)</u>**

Purcell 1956, Nature, 178, 1449

Photon arrival time

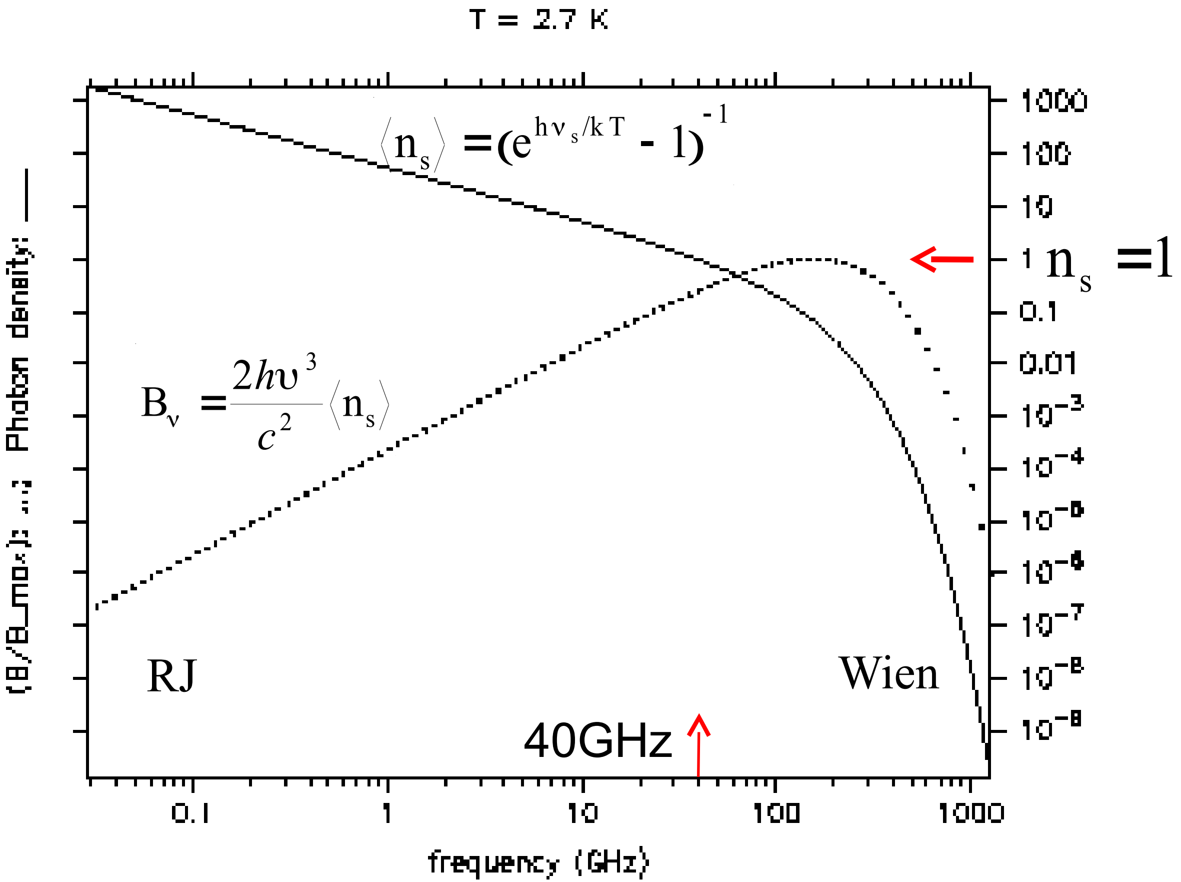
Probability of detecting a second photon after interval t in a beam of linearly polarized light with bandwidth $\Delta\nu$ (Mandel 1963). Exactly the same factor 2 as in Young 2-slit experiment!



Photon arrival times are correlated on timescales ~ $1/\Delta\nu$, leading to fluctuations $\propto$ total flux, ie. fluctuations are amplified by constructive or destructive interference on timescales ~ $1/\Delta\nu$.

When is wave noise important?

- Photon occupation number in CMB (2.7K)
- CMB alone is enough to put radio observations into wave noise regime

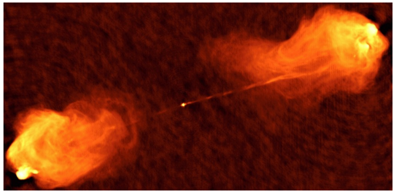


$$\text{Wien: } n_s < 1 \Rightarrow \text{noise} \propto \sqrt{n_s} \quad \text{(counting stats)}$$

$$\text{RJ: } \quad n_s > 1 \Rightarrow \text{noise} \propto n_s \quad \text{(wave noise)}$$

Photon occupation number: examples

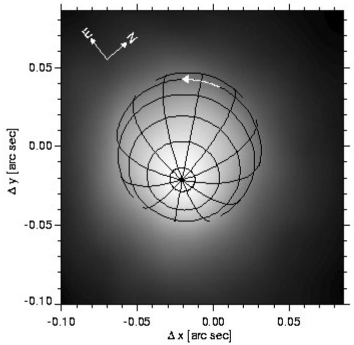
Bright radio source



$$\text{Cygnus A at 1.4GHz at VLA}: T_A = 140\,K \Rightarrow \frac{h\nu}{kT} = 0.0005$$

$$\Rightarrow n_s = (e^{\frac{h\nu}{kT}} - 1)^{-1} = 2000\,Hz^{-1}\,sec^{-1} \therefore \text{wave noise dominated}$$

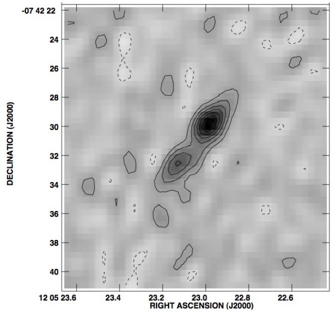
Optical source



$$\text{Betelgeuse resolved by HST}: T_B = 3000K \Rightarrow h\nu/kT = 8$$

$$\Rightarrow n_s = 0.0003\,Hz^{-1}\,sec^{-1} \quad \therefore \text{counting noise dominated}$$

Faint radio source



$$\text{Quasar at } z = 4.7 \text{ with VLA}: S_{1.4GHz} = 0.2\,mJy \;\; (10^{-7} \text{ x CygA})$$

$$T_A = 0.02\,mK \Rightarrow h\nu / kT = 3000 \;\therefore n_s << 1$$

Why do we still assume wave noise dominates in sens. equ?

$$\text{Answer}: \; T_{BG} > 2.7K \text{ ensures } n_s > 1 \text{ always at cm wavelengths.}$$

Night sky is not dark at radio frequencies!

Wave noise: summary

In radio astronomy, the noise statistics are wave noise dominated, ie. noise limit is proportional to the total power, n_s, and not the square root of the power, $n_s^{1/2}$

$$\text{Optical} = \text{Wien}: n_s < 1 \Rightarrow \text{noise} \propto \sqrt{n_s} \;\; \text{(counting stats)}$$

$$\text{Radio} = \text{RJ}: \quad n_s > 1 \Rightarrow \text{noise} \propto n_s \;\; \text{(wave noise)}$$

Concept 2: Quantum noise of coherent amplifiers

Phase conserving electronics => $\Delta\varphi < 1$ rad

Uncertainty principle for photons :

$$\Delta E\,\Delta t = h$$

$$\Delta E = h\nu\,\Delta n_s$$

$$\Delta t = \frac{\Delta\phi}{\nu 2\pi}$$

$$\Rightarrow \Delta\phi\,\Delta n_s = 1\ \text{rad}\ \text{Hz}^{-1}\,\text{sec}^{-1}\ \therefore \Delta\phi < 1\text{rad} \Rightarrow \Delta n_s > 1$$

Phase conserving amplifier has minimum noise: $n_s = 1$ => puts signal into RJ regime = wave noise dominated.

Quantum limit: $$n_s = (e^{h\nu/kT} - 1) = 1$$

$$T_{min} = h\nu/k$$

Quantum noise: Einstein Coefficients and masers

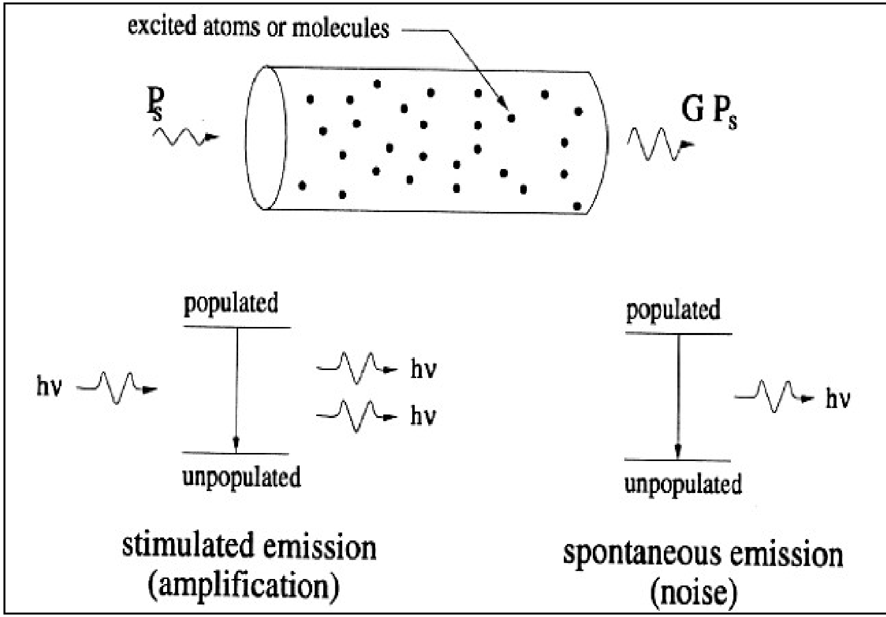
$$\text{Radiative Transfer}: \frac{\partial I}{\partial x} = \frac{h\nu}{c}\left[B_{ij}n_i - B_{ji}n_j\right] I + A_{ij}n_i \frac{h\nu}{4\pi}$$

Rohlfs & Wilson equ 11.8 – 11.13

$$\text{Stimulated emission} = B_{ij}$$

$$\text{Absorption}: g_i B_{ji} = g_j B_{ij}$$

$$\text{Spontaneous emission} = A_{ij} = \frac{8\pi h\nu^3}{c^3} B_{ij}$$



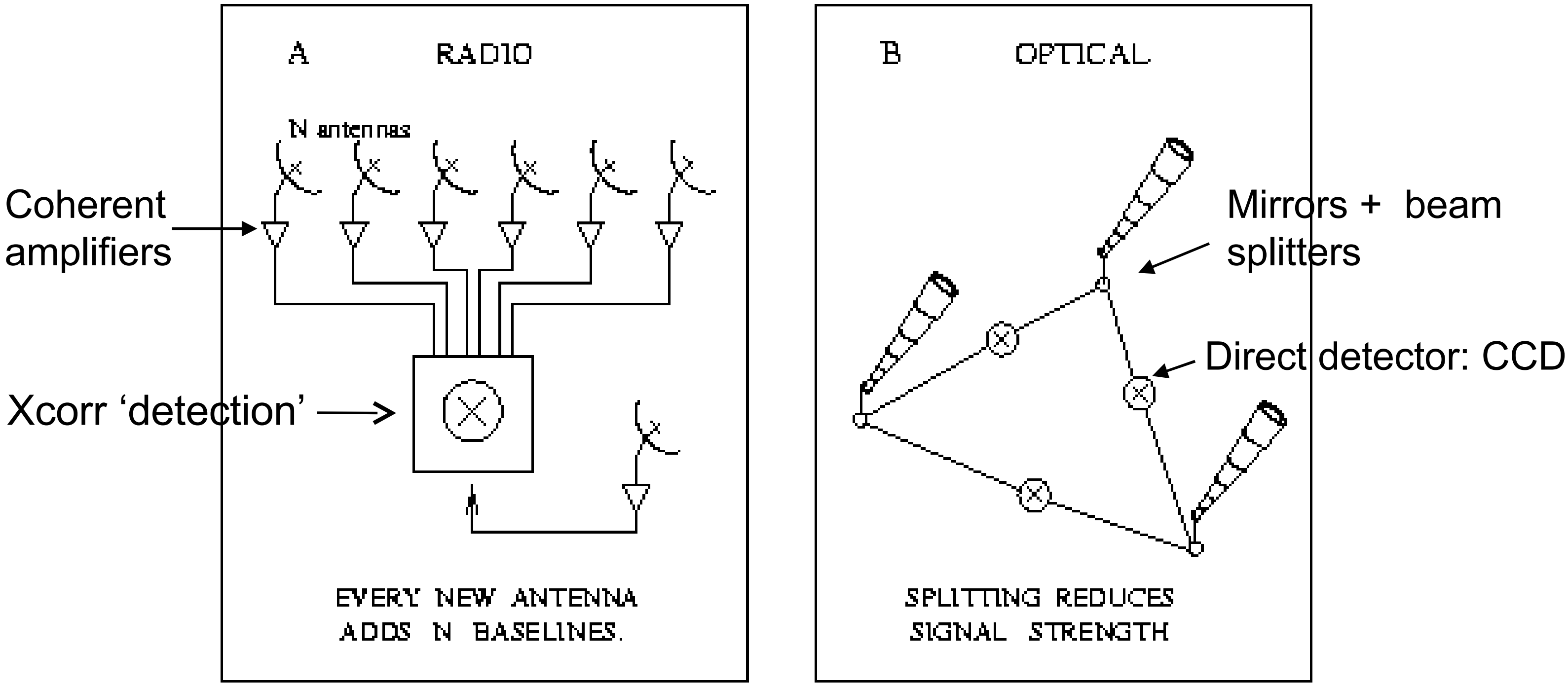
$$\frac{\text{Stimulated}}{\text{Spontaneous}} = \frac{\frac{h\nu}{c} B_{ij} n_i I}{\frac{h\nu}{4\pi} A_{ij} n_i} = \frac{c^2 I}{2h\nu^3}$$

$$I_\nu \rightarrow B_\nu = \frac{2k\nu^2}{c^2} T_B = \frac{2k}{\lambda^2} T_B$$

$$\frac{\text{Stimulated}}{\text{Spontaneous}} = \frac{k\,T_B}{h\nu} \geq 1 \quad \Rightarrow \quad \boxed{T_{min} \geq \frac{h\nu}{k}}$$

Stimluated emission => pay price of spontaneous emission

Consequences: Quantum noise of coherent amplifier ($n_q = 1$)



n_s>>1 => QN irrelevant, use phase coherent amplifiers

Good: adding antennas doesn’t affect SNR per pair, Polarization and VLBI!

Bad: paid QN price

n_s<<1 => QN disaster, use beam splitters, mirrors, and direct detectors

Good: no receiver noise

Bad: adding antenna lowers SNR per pair as N^2

Concept 3: What's all this about temperatures? Johnson-Nyquist electronic noise of a resistor at T_R

JULY, 1928 PHYSICAL REVIEW VOLUME 32

THERMAL AGITATION OF ELECTRICITY IN CONDUCTORS

BY J. B. JOHNSON

JULY, 1928 PHYSICAL REVIEW VOLUME 32

THERMAL AGITATION OF ELECTRIC CHARGE IN CONDUCTORS*

BY H. NYQUIST

THE REVIEW OF SCIENTIFIC INSTRUMENTS VOLUME 17, NUMBER 7 JULY, 1946

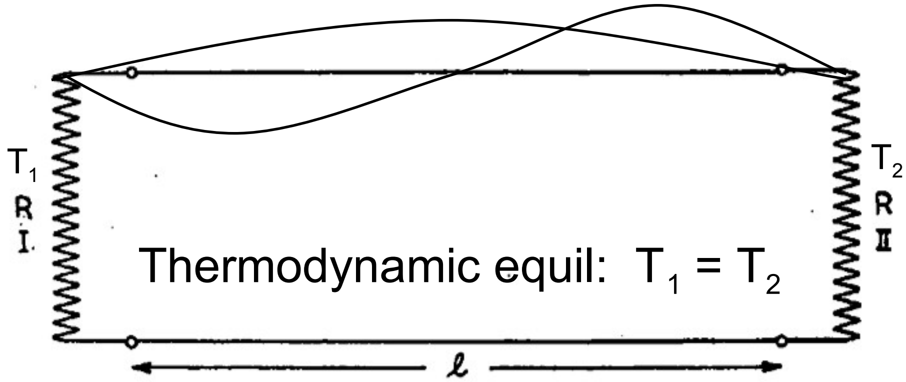
The Measurement of Thermal Radiation at Microwave Frequencies

R. H. DICKE*
*Radiation Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts***

(Received April 15, 1946)

Johnson-Nyquist Noise

$\langle V \rangle = 0$, but $\langle V^2 \rangle \neq 0$



"Statistical fluctuations of electric charge in all conductors produce random variations of the potential between the ends of the conductor…producing mean-square voltage" => white noise power, $\langle V^2 \rangle / R$, radiated by resistor, T_R

Analogy to modes in black body cavity (Dickey):

- Transmission line electric field standing wave modes: $\nu = c/2\ell, 2c/2\ell \ldots Nc/2l$
- # modes (=degree freedom) in $\nu + \Delta\nu$: $\langle N \rangle = 2\ell\, \Delta\nu / c$
- Therm. Equipartion law: energy/degree of freedom: $\langle E \rangle \sim kT$
- Energy equivalent on line in $\Delta\nu$: $E = \langle E \rangle \langle N \rangle = (kT2\ell\Delta\nu) / c$
- Transit time of line: $t \sim \ell / c$
- Noise power transferred from each R to line ~ E/t = $\boxed{P_R = kT_R\, \Delta\nu \text{ erg s}^{-1}}$

Johnson-Nyquist Noise

Thermal noise:

$\langle V^2 \rangle / R$ = white noise power

k_B = 1.27e-16 +/ 0.17 erg/K

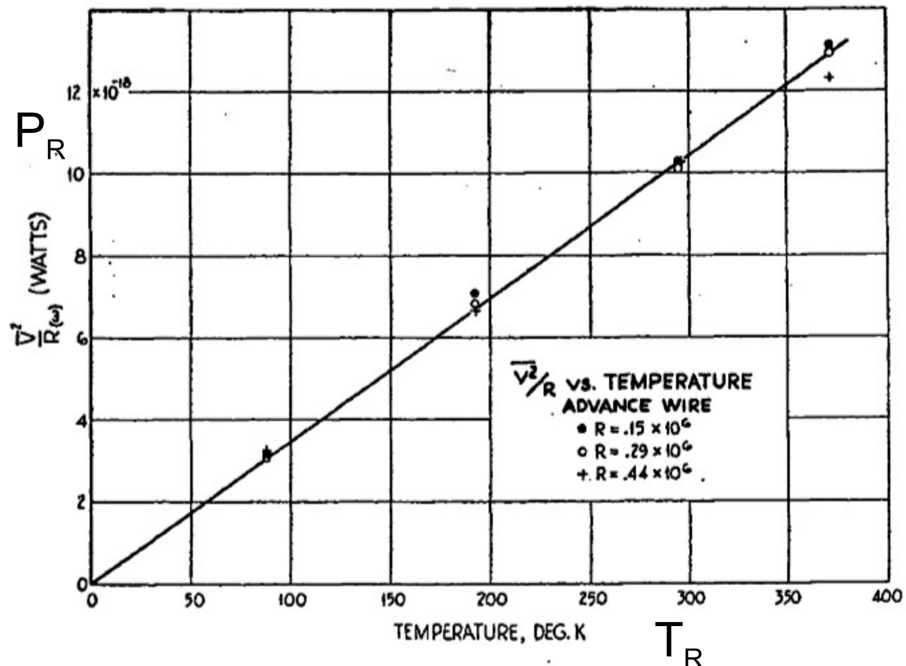


Fig. 6. Apparent power vs. temperature, for Advance wire resistances.

Noise power is strictly function of T_R, not function of R or material.

Antenna Temperature

In radio astronomy, we reference power received from the sky, ground, or electronics, to noise power from a load (resistor) at temperature, T_R = Johnson noise

Consider received power from a cosmic source, P_{src}

- $P_{src} = A_{eff}\, S_\nu\, \Delta\nu$ erg s^{-1}
- Equate to Johnson-Nyquist noise of resistor at T_R: $P_R = kT_R\, \Delta\nu$
- 'equivalent load' due to source = antenna temperature, T_A:

$$kT_A\, \Delta\nu = A_{eff}\, S_\nu\, \Delta\nu \quad \Rightarrow \quad \boxed{T_A = A_{eff}\, S_\nu / k}$$

Brightness Temperature

- Brightness temp = surface brightness (Jy/SR, Jy/beam, Jy/arcsec^2)

- T_B = temp of equivalent black body, B_ν, with surface brightness = source surface brightness at ν: $I_\nu = S_\nu / \Omega = B_\nu = kT_B / \lambda^2$

- $\boxed{T_B = \lambda^2 S_\nu / 2 k \Omega}$

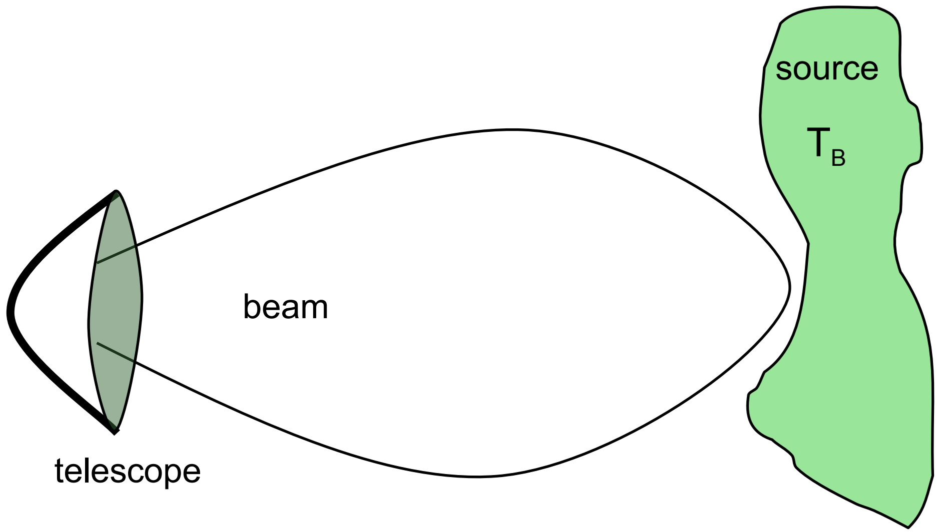
- T_B = physical temperature for optically thick thermal object

- $T_A <= T_B$ always

Source size > beam $T_A = T_B$[a]

Source size < beam $T_A < T_B$

Explains the fact that temperature in focal plane of telescope cannot exceed $T_{B,src}$



[a] Consider 2 coupled horns in equilibrium, and the fact that T_B(horn beam) = T_{sys}

Radiometry and Signal to Noise

Concept 4: number of independent measurements

- Limiting signal-to-noise (SNR): Standard deviation of the mean

$$SNR_{lim} = \frac{\text{Signal}}{\text{Noise per measurement}} \times \sqrt{\#\ \text{of independent measurements}}$$

- Wave noise ($n_s > 1$): noise per measurement = $(\text{variance})^{1/2}$ = $<n_s>$

=> noise per measurement $\propto$ total noise power $\propto T_{sys}$

- Recall, source signal = T_A

$$SNR_{lim} = \frac{T_A}{T_{sys}} \sqrt{\#\ \text{independent measurements}}$$

- Inverting, and dividing by signal, can define 'minimum detectable signal':

$$\Delta T_{lim} = \frac{T_{sys}}{\sqrt{\#\ \text{independent measurements}}}$$

Number of independent measurements

How many independent measurements are made by single interferometer (pair ant) for total time, t, over bandwidth, $\Delta\nu$?

Return to uncertainty relationships:

$\Delta E \Delta t = h$

$\Delta E = h\Delta\nu$

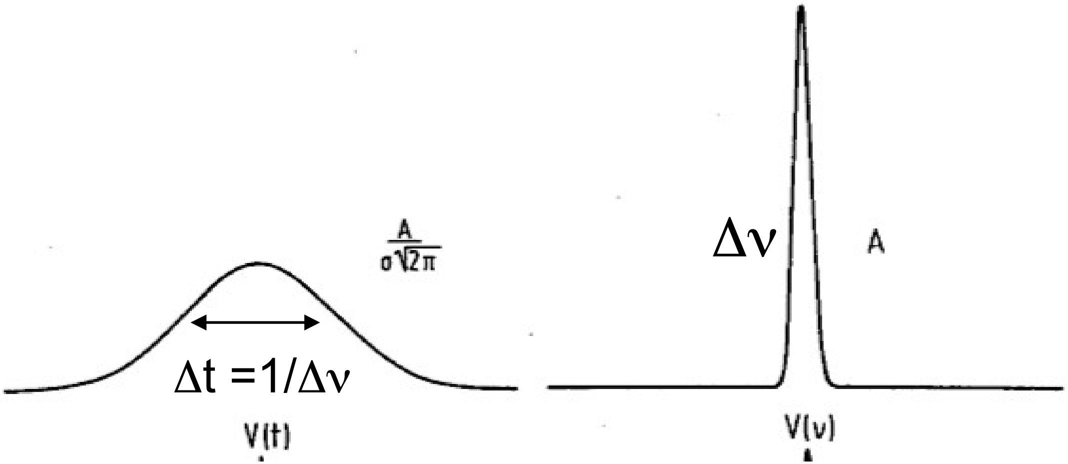
$\Delta\nu\Delta t = 1$

Δt = minimum time for independent measurement = $1/\Delta\nu$[a]

independent measurements in t = $t/\Delta t$ = $\boxed{\mathbf{t\ \Delta\nu}}$

[a]Measurements on shorter timescales provide no new information, eg. consider monochromatic signal => $t \rightarrow \infty$ and single measurement dictates waveform ad infinitum

General Fourier conjugate variable relationships



- Fourier conjugate variables: frequency – time

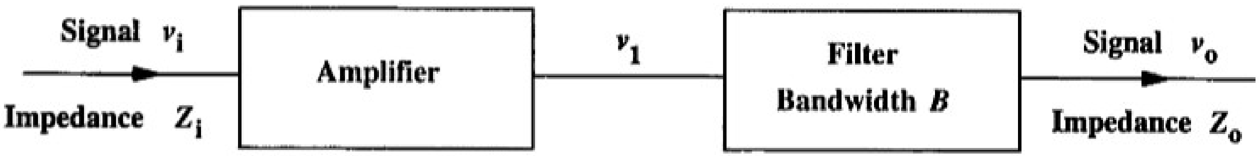
- If V(ν) is Gaussian of width Δν, then V(t) is also Gaussian of width = $\Delta t = 1/\Delta\nu$

- Measurements of V(t) on timescales $\Delta t < 1/\Delta\nu$ are correlated, ie. *not independent*

- Restatement of Nyquist sampling theorem: maximum information on band-limited signal is gained by sampling at $t < 1/2\Delta\nu$. Nothing changes on shorter timescales.

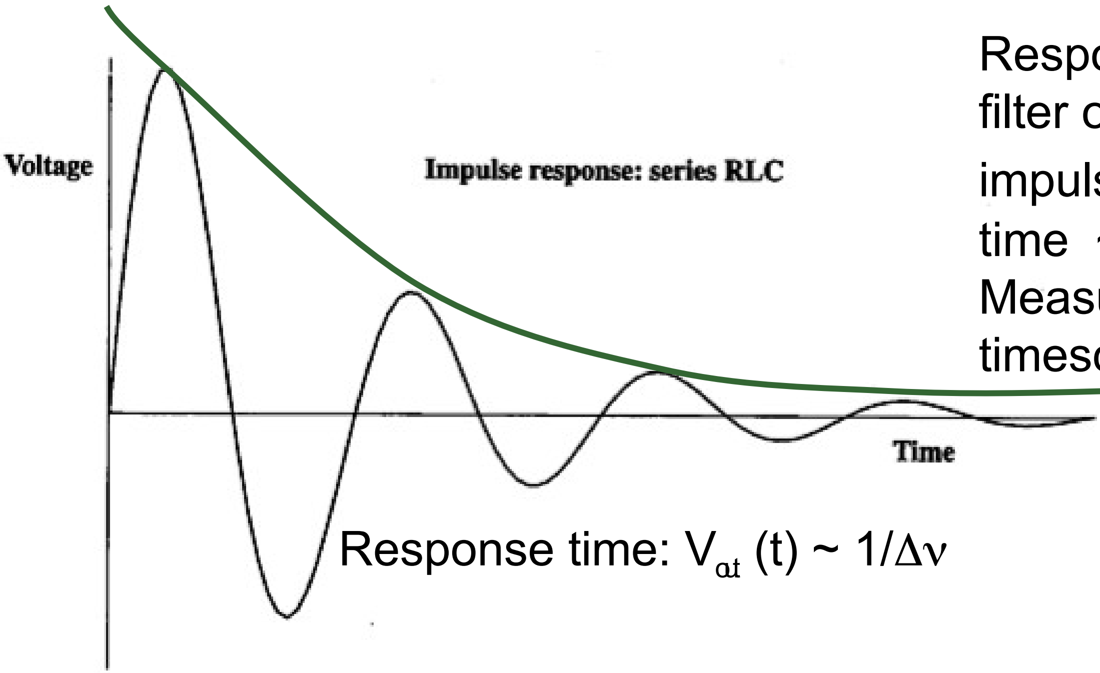
Response time of a bandpass filter

$V_{in}(t) = \delta(t)$



$\Delta\nu$

$V_{\alpha t}(t) \sim 1/\Delta\nu$



Response time: $V_{\alpha t}(t) \sim 1/\Delta\nu$

Fig. 3.2. Impulse response of an RLC filter.

Response of RLC (tuned) filter of bandwidth $\Delta\nu$ to impulse $V(t) = \delta(t)$: decay time $\sim 1/\Delta\nu$. Measurements on shorter timescales are correlated[a].

[a]classical analog' to concept 1 = correlated arrival time of photons

Interferometric Radiometer Equation

Interferometer pair: $$\Delta T_{lim} = \frac{T_{sys}}{\sqrt{\#\ \text{independent measurements}}} = \frac{T_{sys}}{\sqrt{\Delta\nu t}}$$

Antenna temp equation: $\Delta T_A = A_{eff}\ \Delta S_\nu / k$

Sensitivity for interferometer pair: $$\Delta S_{lim} = \frac{kT_{sys}}{A_{eff}\sqrt{\Delta\nu t}}$$

Finally, for an array, the number of independent measurements at give time = number of pairs of antennas = $N_A(N_A-1)/2$

$$\Delta S_{lim} = \frac{kT_{sys}}{A_{eff}\sqrt{N_A(N_A - 1)\Delta\nu t}}$$

Can be generalized easily to: # polarizations, inhomogeneous arrays (A_i, T_i), digital efficiency terms…

Fun with noise: Wave noise vs. counting statistics

- Received source power ~ $S_{src} \times A_{eff}$
- Optical telescopes ($n_s < 1$) => rms ~ $N_\gamma^{1/2}$

$$N_\gamma \propto S_{src} A_{eff} \quad \Rightarrow \text{SNR} = \text{signal/rms} \propto (A_{eff})^{1/2}$$

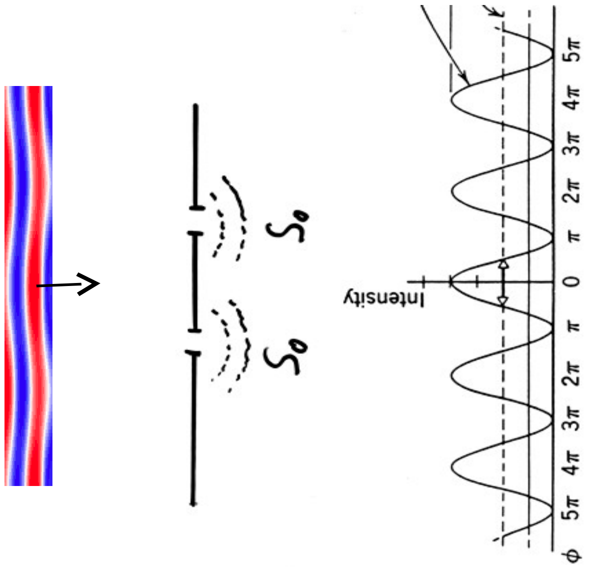
- Radio telescopes ($n_s > 1$) => rms ~ 'N_γ'

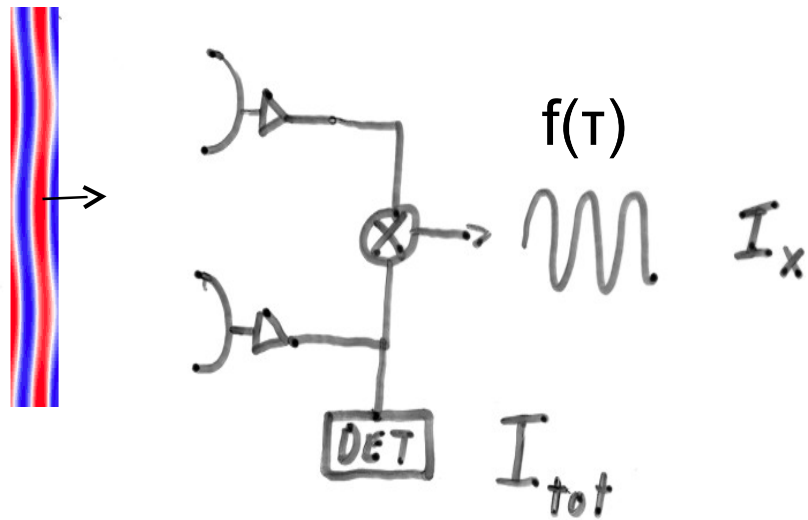
$$'N_\gamma' \propto T_{sys} = T_A + T_{Rx} + T_{BG} + T_{spill}$$

➢ Faint source: $T_A \ll (T_{Rx} + T_{BG} + T_{spill})$ => rms dictated by receiver => SNR $\propto A_{eff}$

➢ Bright source: $T_{sys} \sim T_A \propto S_{src} A_{eff}$ => SNR *independent* of A_{eff}

Quantum noise and the 2 slit paradox: wave-particle duality





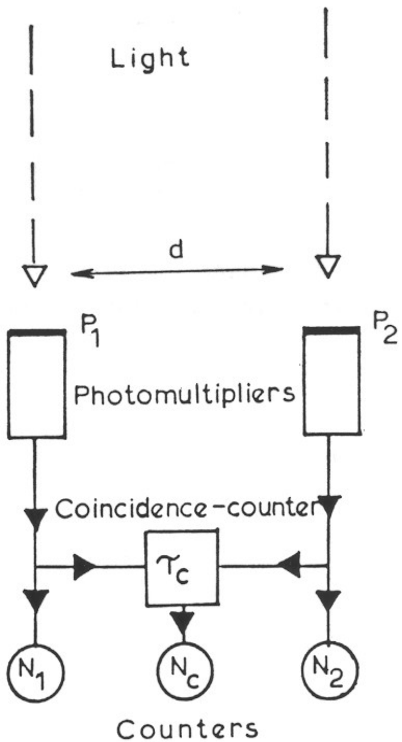
Interference pattern builds up even with photon-counting experiment.

Which slit does the photon enter? With a phase conserving amplifier it seems one could replace slit with amplifier, and both detect the photon and 'build-up' the interference pattern (which we know can't be correct). But quantum noise dictates that the amplifier introduces 1 photon noise, such that:

$$n_s = 1 +/- 1$$

and we still cannot tell which slit the photon came through!

Intensity Interferometry: 'Hanbury-Brown – Twiss Effect' Replace amplifiers with square-law detector ('photon counter'). This destroys phase information, but cross correlation of intensities still results in a finite correlation! Exact same phenomenon as increased correlation for t < $1/\Delta\nu$ in photon arrival time (concept 1), ie. correlation of the wave noise itself.



- Voltages correlate on timescales ~ $1/\nu$, with correlation coef, γ
- Intensities correlate on timescales ~ $1/\Delta\nu$, with correlation coef, γ^2

Advantage: timescale = $1/\Delta\nu$ (not $1/\nu$) => insensitive to poor optics, 'seeing'

Disadvantage: No visibility phase information

Lower SNR

Interferometric Radiometer Equation

$$S_{rms} = \frac{2kT_{sys}}{A_{eff}\sqrt{N_A(N_A - 1)t_{\text{int}}\Delta\nu}}$$

- T_{sys} = wave noise for photons: rms $\propto$ total power
- A_{eff} ,k_B = Johnson-Nyquist noise + antenna temp definition
- $t\Delta\nu$ = # independent measurements of T_A/T_{sys} per pair of antennas
- N_A = # indep. meas. for array


END
ESO

Electron statistics: Fermi-Dirac (indistiguishable particles, but number of particles in each state = 0 or 1, or antisymmetric wave function under particle exchange, spin ½)

$$< n_s >= [e^{(\hbar\omega_s/kT)} + 1]^{-1}$$

$$< \Delta n_s^2 > \; \equiv \; < (n_s - < n_s >)^2 > \; = \; < n_s > - < n_s >^2$$

$$\text{eg. maximum } \langle n_s \rangle = 1 \Rightarrow \text{ all states are filled } \therefore \text{ variance } = 0$$

Quantum limit VI: Heterodyne vs. direct detection interferometry

$$\frac{SNR_{Het}}{SNR_{DD}} \approx \left(N/(e^{h\nu/kT} - 1)\right) = \left(N n_s\right)^{1/2} \left(\frac{\Delta\nu_{Het}}{\Delta\nu_{DD}}\right)^{1/2}, \text{ where } N = \text{number of elements}$$

Betelgeuse with HST : Size $\approx$ diffraction limited beam $\approx$ 40mas

$$T_B = 3000\ K \ => h\nu/kT = 8 \ \text{at}\ 5e14\ Hz$$

$$\frac{SNR_{Het}}{SNR_{DD}} = \left(N/400\right)^{1/2} \Rightarrow \text{DD wins}$$

Cygnus A with VLA at 1.4 GHz :

$$T_A = 140\,K \Rightarrow \frac{h\nu}{kT} = 0.0005$$

$$\frac{SNR_{Het}}{SNR_{DD}} = \left(2000N\right)^{1/2} \Rightarrow \text{Heterodyne wins}$$

Orion at 350 GHz (200 Jy) :

$$T_A = 10\,K \Rightarrow \frac{h\nu}{kT} = 2$$

$$\frac{SNR_{Het}}{SNR_{DD}} = \left(N/4\right)^{1/2} \Rightarrow \text{Toss - up}$$