Sensitivity and Noise

Chris Carilli (NRAO)

SIRA II -- Parts of lectures 9,33,28

Rolfs & Wilson

Burke & Graham Smith

Thomson, Moran, Swenson

Eleventh Synthesis Imaging Workshop

Socorro, June 10-17, 2008

Some definitions

I_ν (or B_ν) = Surface Brightness: erg/s/cm^2/Hz/sr

(= intensity)

S_ν = Flux density: erg/s/cm^2/Hz $\int I_\nu \Delta\Omega$

S = Flux: erg/s/cm^2 $\int I_\nu \Delta\Omega \Delta\nu$

P = Power received: erg/s $\int I_\nu \Delta\Omega \Delta\nu \Delta A_{tel}$

E = Energy: erg $\int I_\nu \Delta\Omega \Delta\nu \Delta A_{tel} \Delta t$

Interferometric Radiometry Equation

$$S_{rms} = \frac{2kT_{sys}}{A_{eff}\sqrt{N_A(N_A - 1)t_{\text{int}}\Delta\nu}}$$

Physically motivate terms

- Wave noise and photon statistics
- Quantum noise (Optical vs. Radio interferometry)
- Temperature in Radio Astronomy (Johnson-Nyquist resistor noise, Antenna Temp, Brightness Temp)
- Number of independent measurements of T_A/T_{sys}
- Some interesting consequences

<u>Photon statistics</u>: Bose-Einstein statistics for gas without number conservation (Reif Chap 9)

Thermal equilibrium => **Planck distribution function**

n_s = photon occupation number, relative number in state s

= number of photons in standing-wave mode in box at temperature T

= number of photons/s/Hz in (diffraction limited) beam in free space (Richards 1994, J.Appl.Phys, 76, 1)

$$\langle n_s \rangle = \left(e^{h\nu_s/kT} - 1\right)^{-1}$$

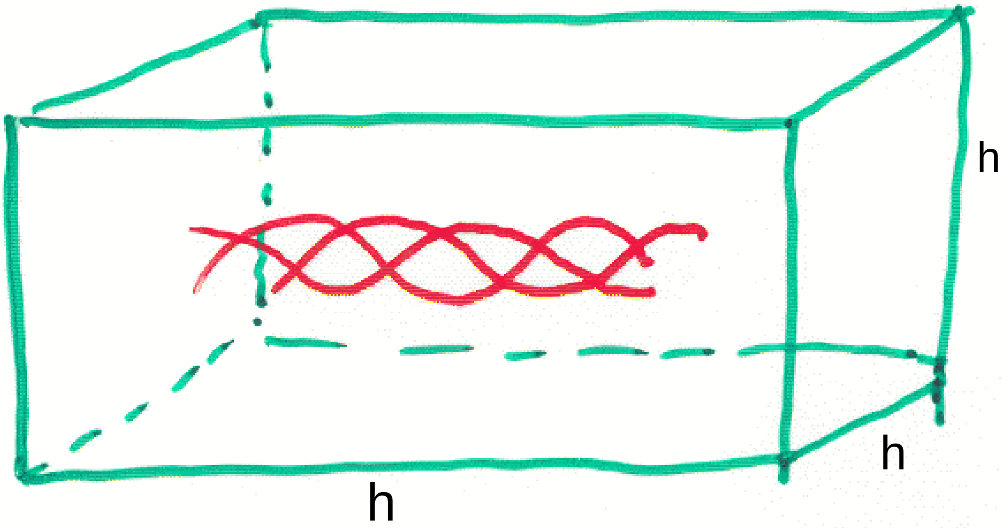
<u>Photon noise</u>: variance in # photons arriving each second in free space beam

$$\langle \Delta n_s^2 \rangle \equiv \left\langle \left(n_s - \langle n_s \rangle\right)^2 \right\rangle = \langle n_s \rangle + \langle n_s \rangle^2$$

$$\langle n_s \rangle = \text{Poisson Stats} = \text{shot noise} = \text{counting stats (root n)}$$

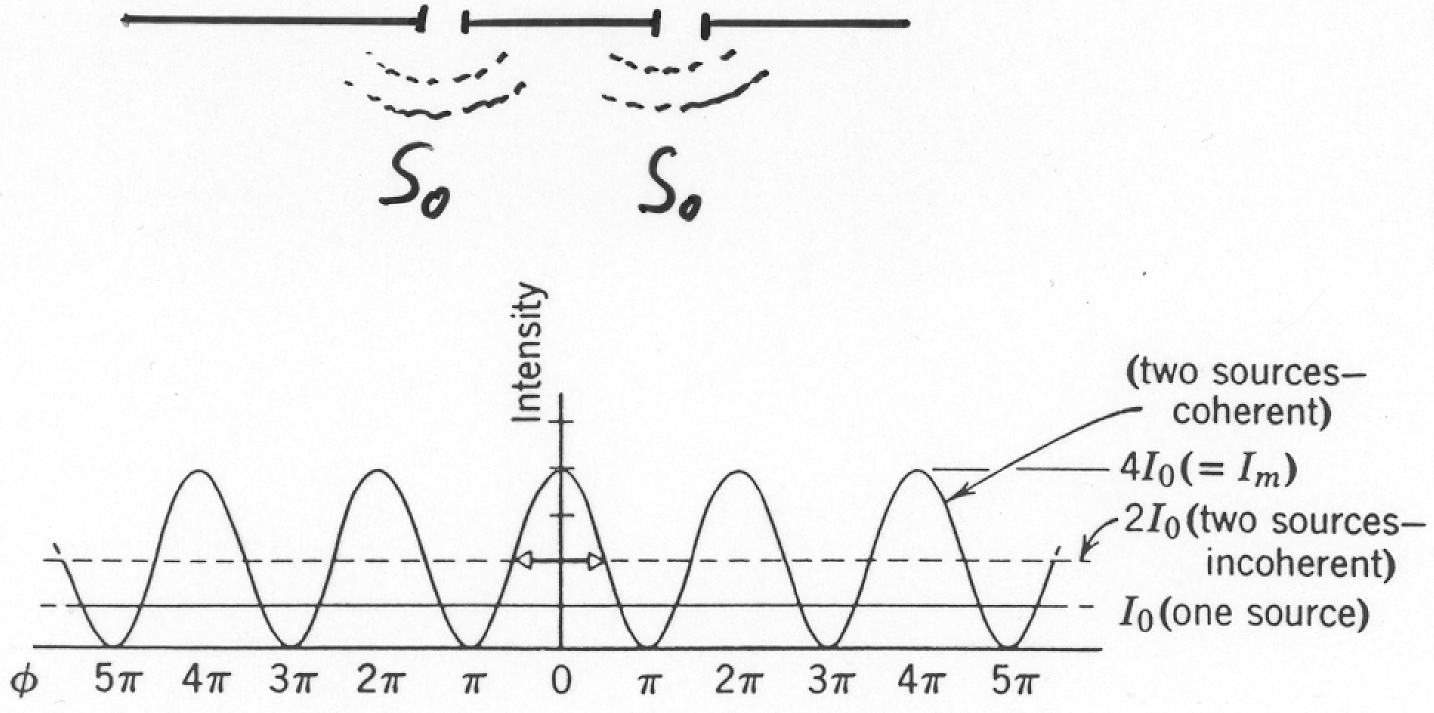
$$\langle n_s^2 \rangle = \text{Wave noise}$$

Origin of wave noise: ‘Bunching of Bosons’ in phase space (time and frequency) allows for interference (ie. coherence).



Bosons can, and will, occupy the exact same phase space if allowed, such that interference (destructive or constructive) will occur. Restricting phase space (ie. narrowing the bandwidth and sampling time) leads to interference within the beam. **This naturally leads to fluctuations that are proportional to intensity (= wave noise).**

Origin of wave noise: coherence -- Young's 2 slit experiment



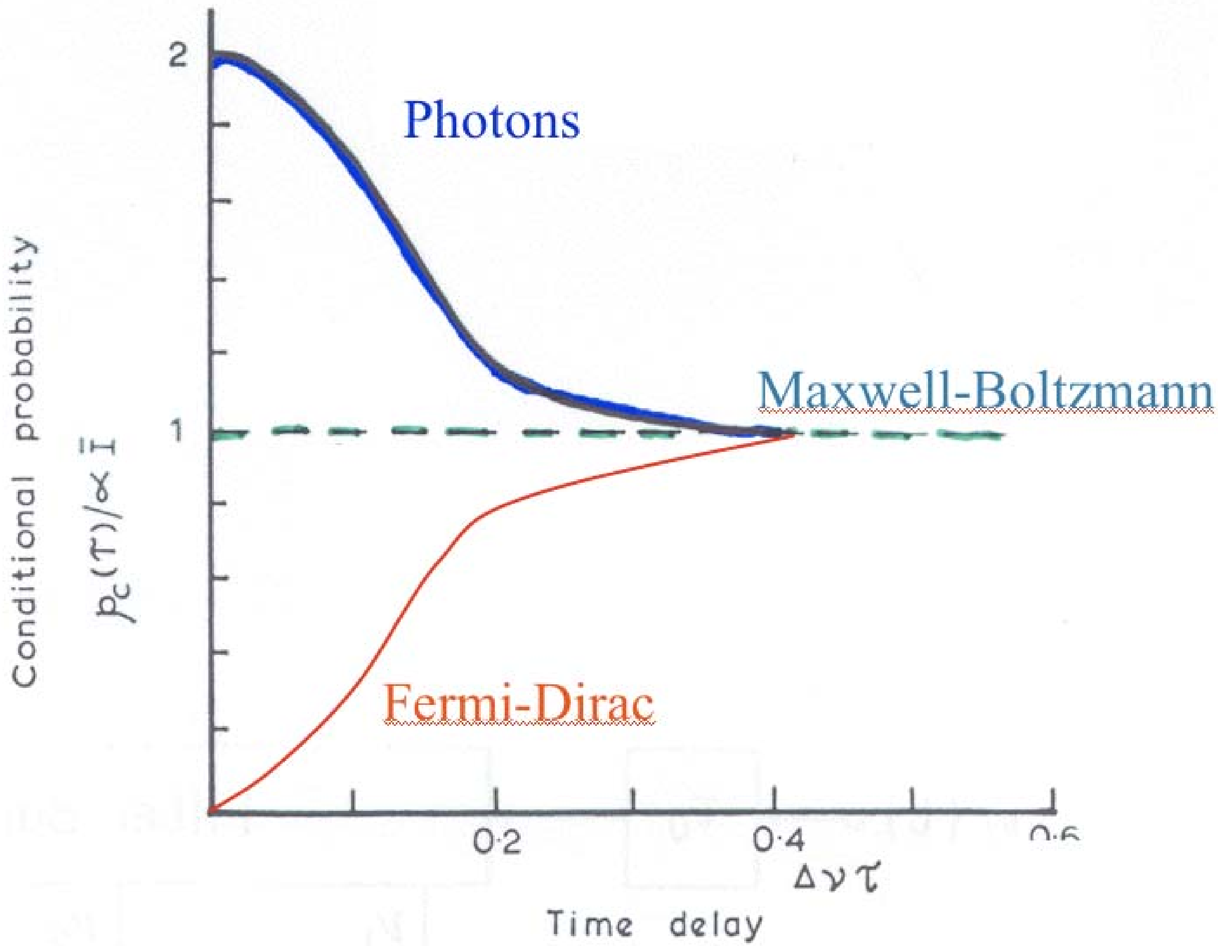
$$\text{Single Source}: I \propto V^2 = \text{'1 photon'}$$

$$\text{Two incoherent sources}: I \propto 2\left(V^2\right) = \text{'2 photons'}$$

$$\text{Two coherent sources}: I \propto (2V)^2 = \text{'0 to 4 photons'}$$

Origin of wave noise

Photon arrival time: normalized probability of detecting a second photoelectron after interval t in a plane wave of linearly polarized light with Gaussian spectral profile of width Δν (Mandel 1963). Exactly the same factor 2 as in Young's slits!



Photon arrival times are correlated on timescales ~ 1/ Δν, which naturally leads to fluctuations in the signal ∝ total flux, ie. fluctuations are amplified by constructive or destructive interference on timescales ~ 1/ Δν.

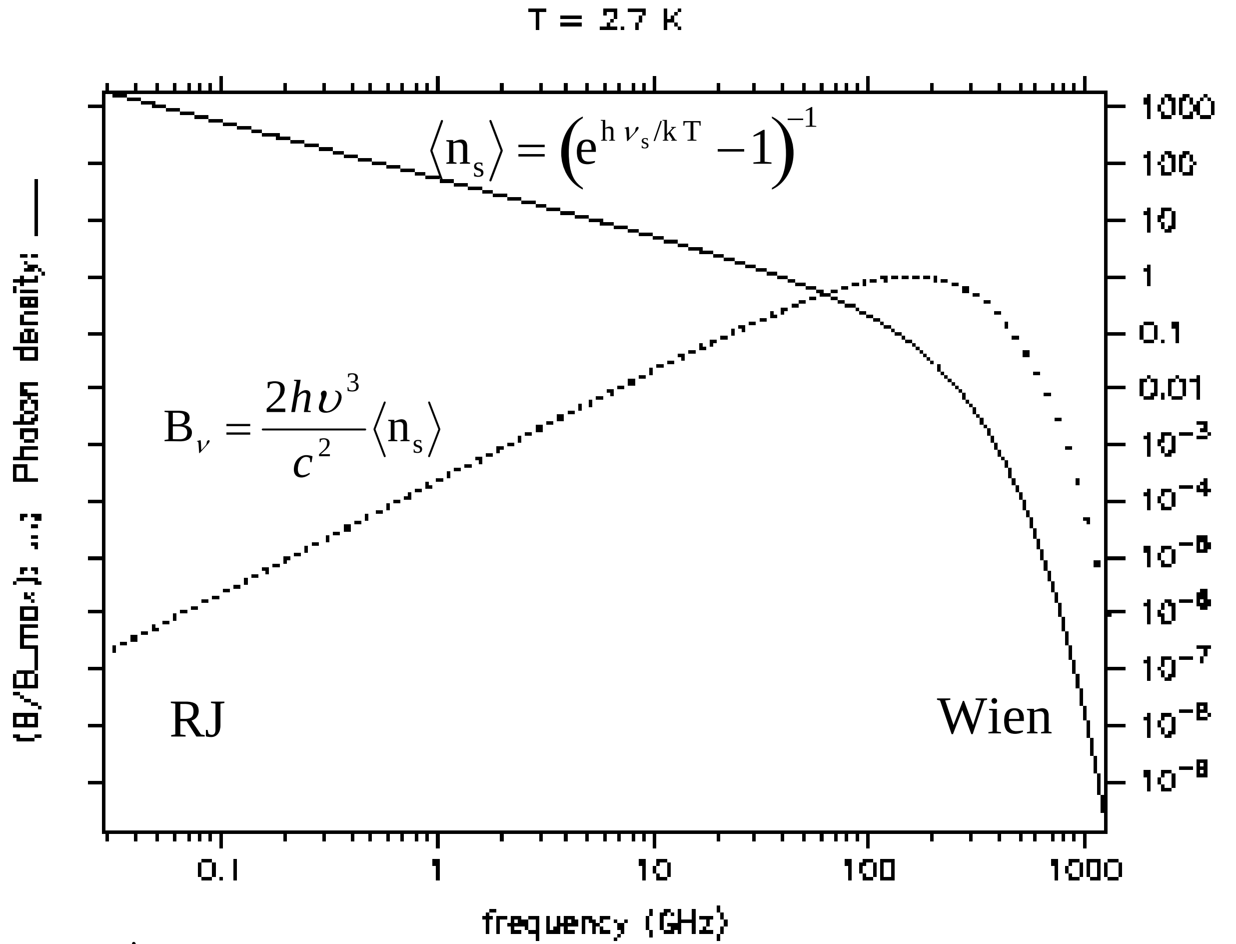
Origin of wave noise III

"Think then, of a stream of wave packets each about c/$\Delta\nu$ long, in a random sequence. There is a certain probability that two such trains accidentally overlap. When this occurs they interfere and one may find four photons, or none, or something in between as a result. It is proper to speak of interference in this situation because the conditions of the experiment are just such as will ensure that these photons are in the same quantum state. **To such interference one may ascribe the 'abnormal' density fluctuations in any assemblage of bosons.**

Were we to carry out a similar experiment with a beam of electrons we should find a suppression of the normal fluctuations instead of an enhancement. The accidental overlapping wave trains are precisely the configurations excluded by the Pauli principle."

Purcell 1956, Nature, 178, 1449

When is wave noise important? Photon occupation number at 2.7K



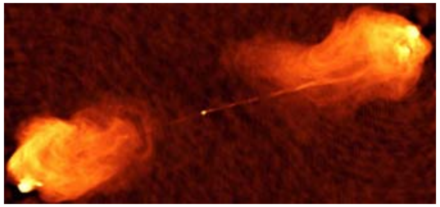
$$\langle \Delta n_s^2 \rangle \equiv \left\langle \left(n_s - \langle n_s \rangle\right)^2 \right\rangle = \langle n_s \rangle + \langle n_s \rangle^2$$

$$\text{Wien: } n_s < 1 \Rightarrow \text{rms} \propto \sqrt{n_s} \text{ (counting stats)}$$

$$\text{RJ: } \quad n_s > 1 \Rightarrow \text{rms} \propto n_s \quad \text{(wave noise)}$$

Photon occupation number: examples

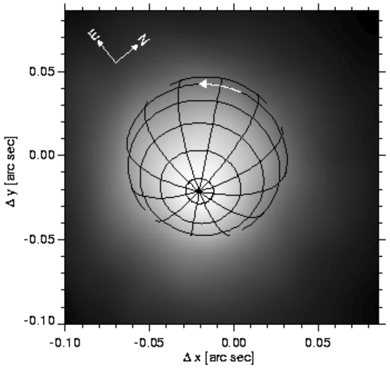
Bright radio source



$$\text{Cygnus A at 1.4GHz at VLA}: T_A = 140\,K \Rightarrow \frac{h\nu}{kT} = 0.0005$$

$$\Rightarrow n_s = (e^{\frac{h\nu}{kT}} - 1)^{-1} = 2000\,Hz^{-1}\,sec^{-1} \therefore \text{wave noise dominated}$$

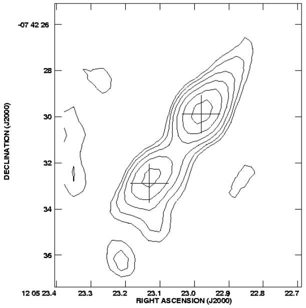
Optical source



$$\text{Betelgeuse resolved by HST}: T_B = 3000K \Rightarrow h\nu/kT = 8$$

$$\Rightarrow n_s = 0.0003\,Hz^{-1}\,sec^{-1} \quad \therefore \text{'counting noise' dominated}$$

Faint radio source



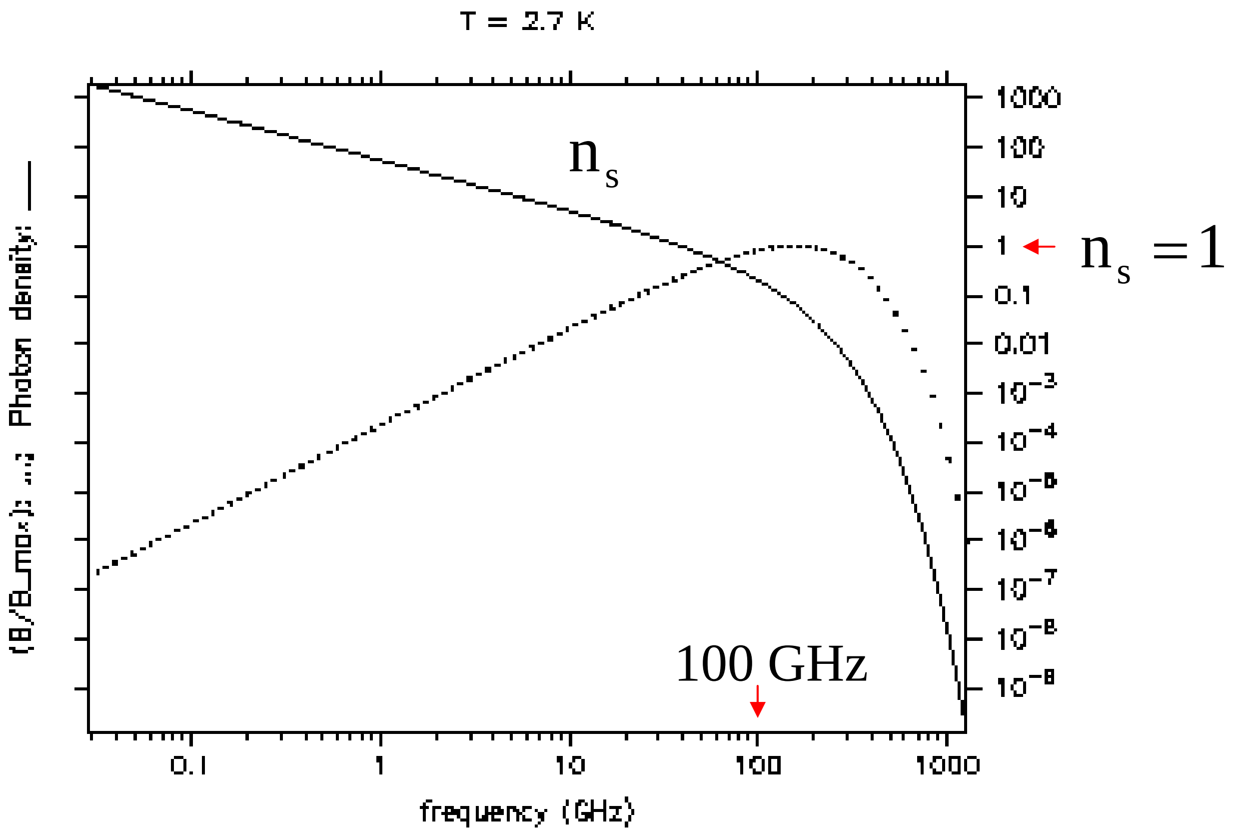
$$\text{Quasar at } z = 4.7 \text{ with VLA}: S_{1.4GHz} = 0.2\,mJy \;\; (10^{-7} \text{ x CygA})$$

$$T_A = 0.02\,mK \Rightarrow h\nu/kT = 3000 \;\therefore n_s << 1$$

Why do we still assume wave noise dominates in sens. equ?

Answer: $T_{BG} > 2.7K$ ensures $n_s > 1$ always at cm wavelengths.

The sky is not dark in the radio!



"Even the feeble microwave background ensures that the occupation number at most radio frequencies is already high. In other words, even though the particular contribution to the signal that we seek is very very weak, it is already in a classical sea of noise and if there are benefits to be derived from retaining the associated aspects, we would be foolish to pass them up." Radhakrishnan 1998

Wave noise: conclusions

In radio astronomy, the noise statistics are wave noise dominated, ie. rms fluctuations are proportional to the total power (n_s), and not the square root of the power ($n_s^{1/2}$)

Noise limit: quantum noise and coherent amplifiers

Uncertainty principle for photons:

$$\Delta E \, \Delta t = h$$

$$\Delta E = h \nu \, \Delta n_s$$

$$\Delta t = \frac{\Delta\varphi}{\nu 2\pi}$$

$$\Rightarrow \Delta\varphi \Delta n_s = 1 \text{ rad Hz}^{-1}\,\text{sec}^{-1}$$

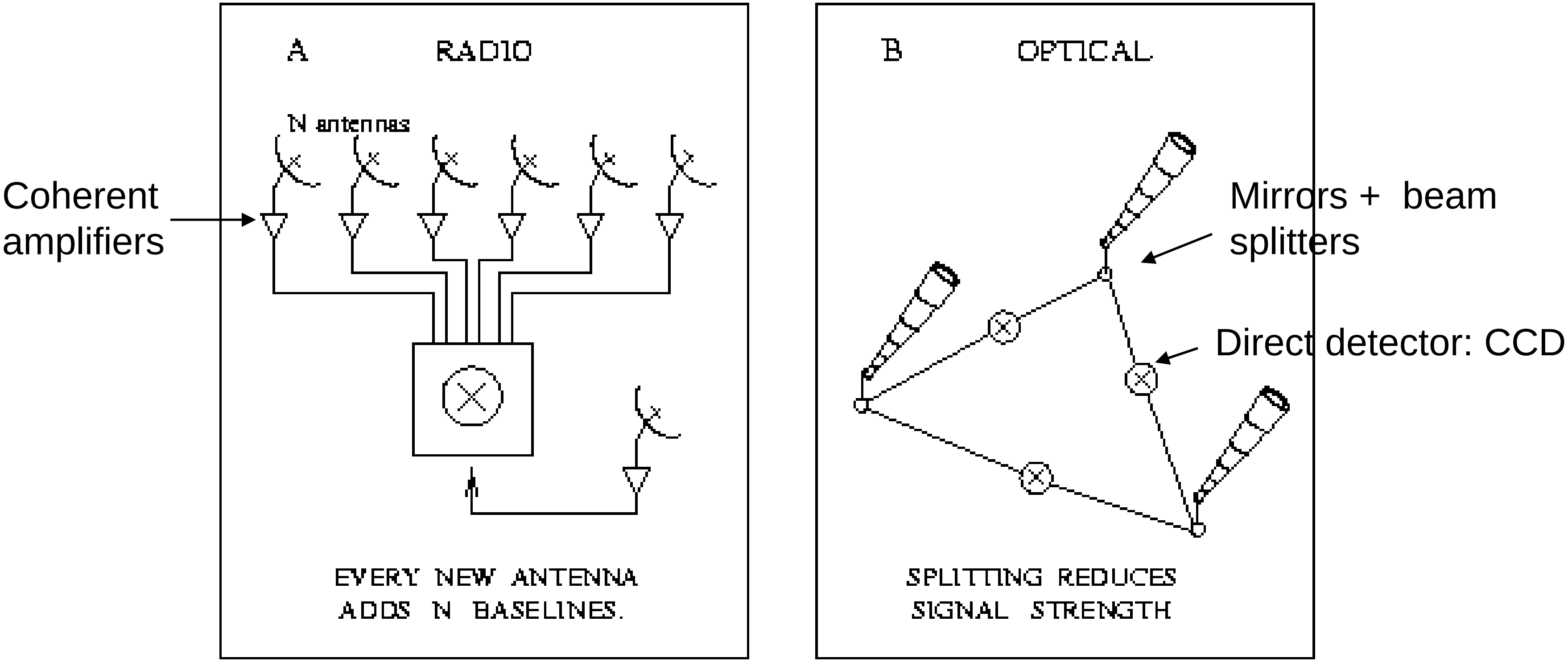
Coherent Amplifier: $\Delta\varphi < 1\text{rad} \Rightarrow \Delta n_s = 1$ photon $\text{Hz}^{-1}\,\text{sec}^{-1}$

Phase conherent amplifier has minimum noise of $n_s = 1$ photon $\text{Hz}^{-1}\,\text{sec}^{-1}$

Phase coherent amplifier automatically puts signal into RJ regime => wave noise dominated

Note: phase coherent amplifier is not a detector

Quantum noise of coherent amplifier: n_q = 1 Hz^{-1} s^{-1}



n_s>>1 => QN irrelevant, use phase conserving electrons

Adv: adding antennas doesn't affect SNR per pair

Disadv: paid QN price

n_s<<1 => QN disaster, use beam splitters, mirrors, and direct detectors

Adv: no receiver noise

Disadv: adding antenna lowers SNR per pair as N^2

Quantum noise: Einstein Coefficients (eg. masers)

$$\text{Stimulated emission} = B_{ij} \qquad \text{Spontaneous emission} = A_{ij} = \frac{8\pi\nu^3 h}{c^3} B_{ij}$$

$$\text{Stimulated Absorption}: \; B_{ij} = B_{ji}$$

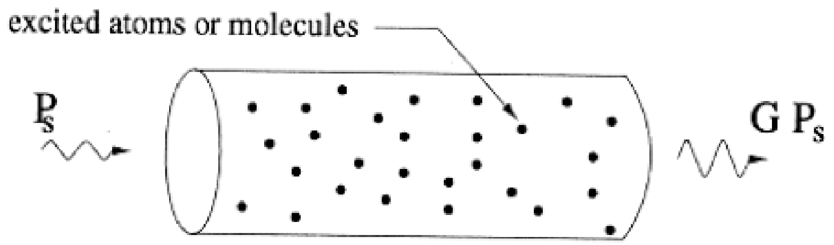
$$\text{Radiative Transfer}: \frac{\partial I}{\partial x} = \frac{h\nu}{c\Delta\nu}\left[B_{ij}n_i - B_{ji}n_j\right] I + A_{ij}n_i \frac{h\nu}{4\pi\Delta\nu}$$

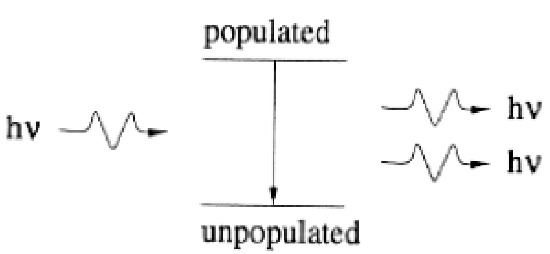
$$\frac{\text{Stimulated}}{\text{Spontaneous}} = \frac{\frac{h\nu}{c\Delta\nu} B_{ij} n_i I}{\frac{h\nu}{4\pi\Delta\nu} A_{ij} n_i} = \frac{c^2 I}{2h\nu^3}$$

$$I_\nu \rightarrow B_\nu = \frac{2k\nu^2}{c^2} T_B = \frac{2k}{\lambda^2} T_B$$

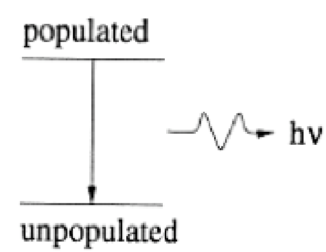
$$\frac{\text{Stimulated}}{\text{Spontaneous}} = \frac{k T_B}{h\nu}$$

$$\Rightarrow \quad T_{min} = \frac{h\nu}{k}$$





stimulated emission
(amplification)



spontaneous emission
(noise)

What's all this about temperatures? Johnson-Nyquist electronic noise of a resistor at T_R

JULY, 1928 PHYSICAL REVIEW VOLUME 32

THERMAL AGITATION OF ELECTRICITY IN CONDUCTORS

By J. B. Johnson

JULY, 1928 PHYSICAL REVIEW VOLUME 32

THERMAL AGITATION OF ELECTRIC CHARGE IN CONDUCTORS*

By H. Nyquist

THE REVIEW OF SCIENTIFIC INSTRUMENTS VOLUME 17, NUMBER 7 JULY, 1946

The Measurement of Thermal Radiation at Microwave Frequencies

R. H. Dicke*
*Radiation Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts***

(Received April 15, 1946)

Johnson-Nyquist Noise

$\langle V \rangle = 0$, but $\langle V^2 \rangle \neq 0$

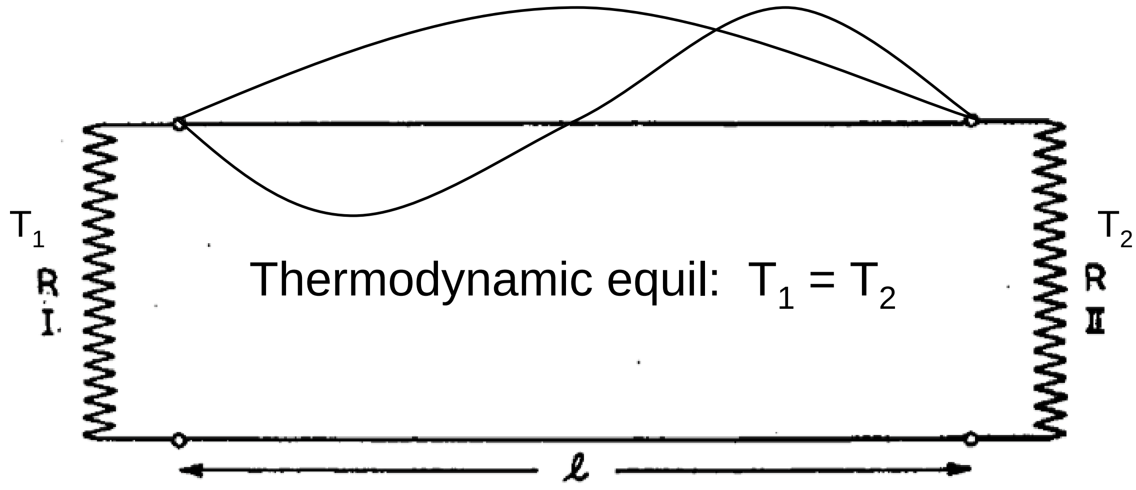


Fig. 3.

"Statistical fluctuations of electric charge in all conductors produce random variations of the potential between the ends of the conductor…producing mean-square voltage" => white noise power, $\langle V^2 \rangle / R$, radiated from resistor at T_R

- Transmission line electric field standing wave modes: $\nu = c/2l, 2c/2l \ldots Nc/2l \ldots$
- # modes (=degree freedom) in $\nu + \Delta\nu$: $\Delta N = 2l\, \Delta\nu / c$
- Therm. Equipartion law: energy/degree of freedom: $\Delta E = h\nu/(e^{h\nu/kT} - 1) \sim kT$ (RJ)
- Energy equivalent on line in $\Delta\nu$: $E = \Delta E\, \Delta N = (kT2l\Delta\nu) / c$
- Transit time of line: $t \sim l / c$
- average power transferred from each R to line in $\Delta\nu \sim E/t =$ $P_R = \mathbf{kT_R}\, \Delta\nu$

Johnson-Nyquist Noise

Thermal noise:

$\langle V^2 \rangle / R$ = 'white noise power'

k_B = 1.27 +/ 0.17 erg/K

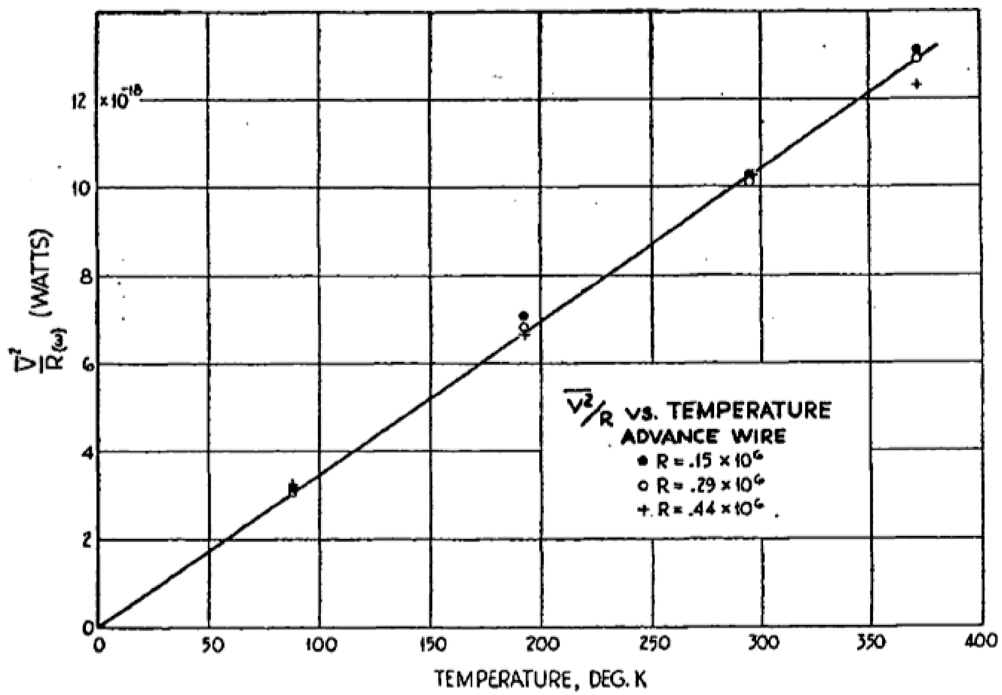


Fig. 6. Apparent power vs. temperature, for Advance wire resistances.

- Noise power is strictly function of T_R, not function of R or material…
- Dickey shows direct analogy with thermal radiation from Black Body
- Nyquist shows direct analogy with thermal motions of molecules in a gas

Antenna Temperature

In radio astronomy, we reference power received from the sky, ground, or electronics, to noise power from a load (resistor) at temperature, T_R = Johnson noise

Consider received power from a cosmic source, P_{src}

- $P_{src} = A_{eff} S_\nu \Delta\nu$ erg s^{-1}
- Equate to Johnson-Nyquist noise of resistor at T_R: $P_R = kT_R \Delta\nu$
- 'equivalent load' due to source = antenna temperature, T_A:

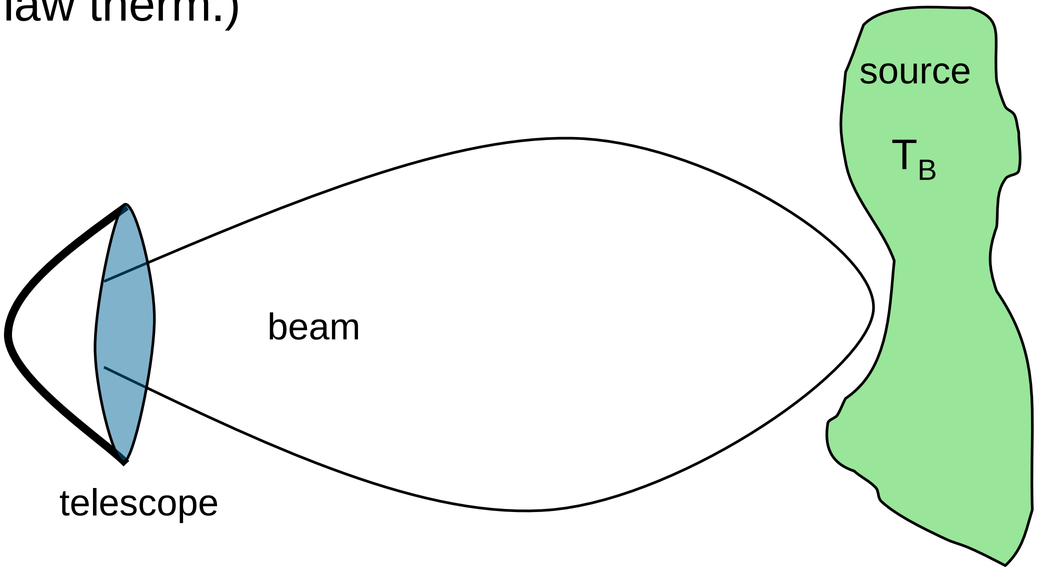
$$kT_A \Delta\nu = A_{eff} S_\nu \Delta\nu \quad \Rightarrow \quad \boxed{T_A = A_{eff} S_\nu / k}$$

Brightness Temperature

- Brightness temp = measure of surface brightness (Jy/SR, Jy/beam, Jy/arcsec^2)
- T_B = temp of equivalent black body, B_ν, with surface brightness = source surface brightness at ν: $I_\nu = S_\nu / \Omega = B_\nu = kT_B / \lambda^2$
- $T_B = \lambda^2 S_\nu / 2 k \Omega$
- T_B = physical temperature for optically thick thermal object
- $T_A <= T_B$ always

Source size > beam $T_A = T_B$ (2nd law therm.)

Source size < beam $T_A < T_B$



[Explains the fact that temperature in focal plane of optical telescope cannot exceed T_B of a source]

Signal to noise and radiometry

- Limiting signal-to-noise (SNR): Standard deviation of the mean

$$\mathrm{SNR_{lim}} = \frac{\mathrm{Signal}}{\mathrm{Noise\ per\ measurement}} \times \sqrt{\#\ \mathrm{of\ independent\ measurements}}$$

- Wave noise ($n_s > 1$): noise per measurement = $(\mathrm{variance})^{1/2} = \langle n_s \rangle$

=> noise per measurement $\propto$ total power noise $\propto T_{sys}$

- Recall, source signal = T_A

$$\mathrm{SNR_{lim}} = \frac{T_A}{T_{sys}} \sqrt{\#\ \mathrm{independent\ measurements}}$$

- Or, inverting, and dividing by signal, can define 'noise' limit as:

$$\Delta T_{lim} = \frac{T_{sys}}{\sqrt{\#\ \mathrm{independent\ measurements}}}$$

Number of independent measurements

How many independent measurements are made by single interferometer (pair ant) for total time, t, over bandwidth, $\Delta\nu$?

Return to uncertainty relationships:

$\Delta E \Delta t = h$

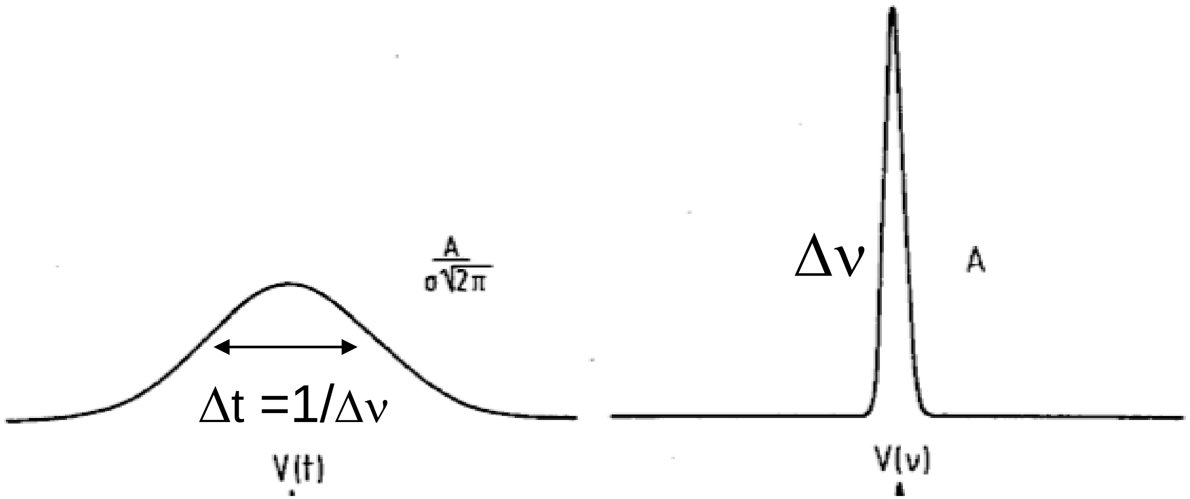
$\Delta E = h\Delta\nu$

$\Delta\nu\Delta t = 1$

Δt = minimum time for independent measurement = $1/\Delta\nu$

independent measurements in t = $t/\Delta t$ = **$t\,\Delta\nu$**

General Fourier conjugate variable relationships



•Fourier conjugate variables, frequency -- time (or power spectrum in freq, autocorrelation in lag, eg. Weiner-Khinchin theorem)

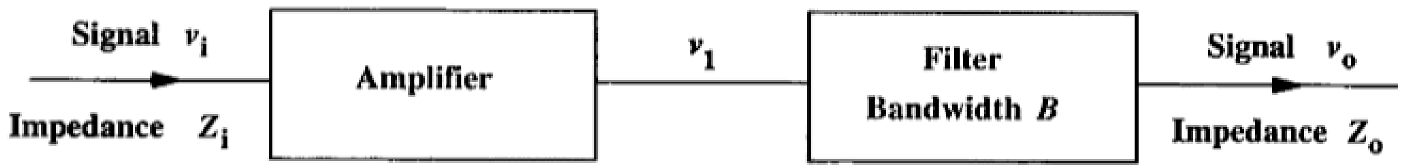
•If V(ν) is Gaussian of width Δν, then V(t) is also Gaussian of width = Δt = 1/Δν

• **Measurements of V(t) on timescales Δt < 1/Δν are correlated, ie. *not independent***

• **Restatement of Nyquist sampling theorem: maximum information is gained by sampling at ~ 1/ 2Δν. Nothing changes on shorter timescales.**

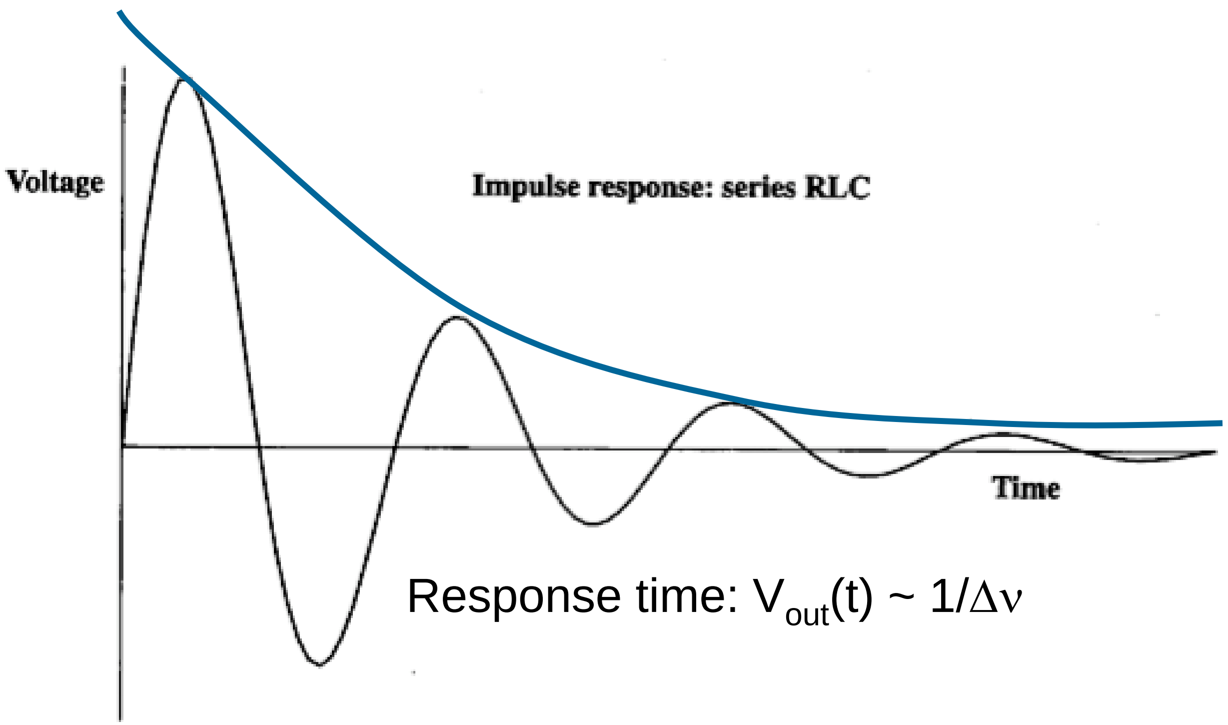
Response time of a bandpass filter

$V_{in}(t) = \delta(t)$



$\Delta\nu$

$V_{out}(t) \sim 1/\Delta\nu$



Response of RLC (tuned) filter of bandwidth $\Delta\nu$ to impulse $V(t) = \delta(t)$: decay time (‘ringing’) ~ $1/\Delta\nu$

Fig. 3.2. Impulse response of an RLC filter.

Interferometric Radiometer Equation

Interferometer pair: $$\Delta T_{\mathrm{lim}} = \frac{T_{\mathrm{sys}}}{\sqrt{\Delta \nu t}}$$

Antenna temp equation: $\Delta T_A = A_{\mathrm{eff}}\ \Delta S_\nu / k$

Sensitivity for single interferometer: $$\Delta S_{\mathrm{lim}} = \frac{kT_{\mathrm{sys}}}{A_{\mathrm{eff}}\sqrt{\Delta \nu t}}$$

Finally, for an array, the number of independent measurements at give time = number of pairs of antennas = $N_A(N_A-1)/2$

$$\Delta S_{\mathrm{lim}} = \frac{kT_{\mathrm{sys}}}{A_{\mathrm{eff}}\sqrt{N_A(N_A-1)\Delta \nu t}}$$

Can be generalized easily to: # polarizations, inhomogeneous arrays (A_i, T_i), digital efficiency terms…

Fun with noise: Wave noise vs. counting statistics

- Received source power $\propto$ telescope area = A_{eff}

- Optical telescopes: $n_s < 1 \Rightarrow$ rms ~ $n_s^{1/2}$

$n_s \propto A_{eff} \Rightarrow$ SNR = signal/rms $\propto (A_{eff})^{1/2}$

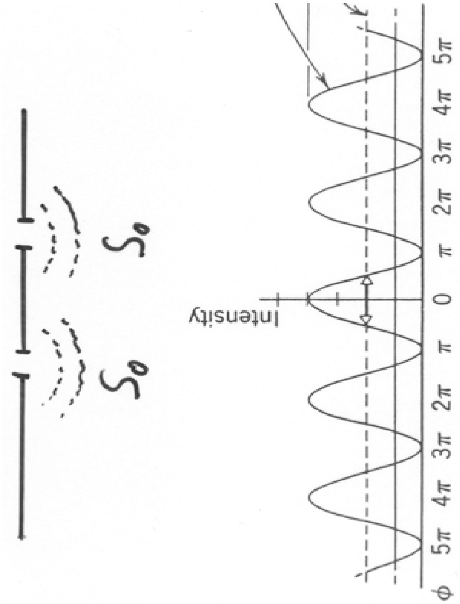
- Radio telescopes: $n_s > 1 \Rightarrow$ rms ~ n_s

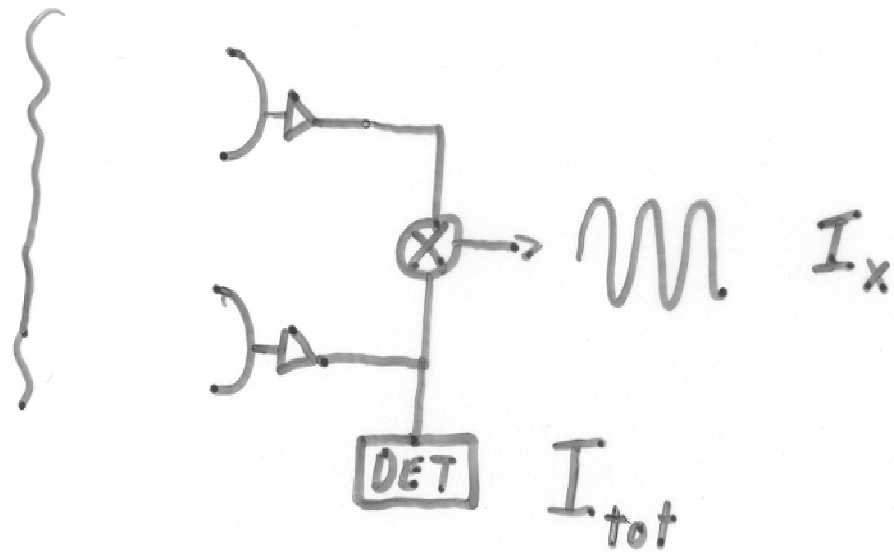
$n_s \propto T_{sys} = T_{Rx} + T_A + T_{BG} + T_{spill}$

➢ Faint source: $T_A \ll (T_{Rx} + T_{BG} + T_{spill}) \Rightarrow$ rms dictated completely by receiver (independent of A_{eff}) $\Rightarrow$ SNR $\propto A_{eff}$

➢ Bright source: $T_{sys} \sim T_A \propto A_{eff} \Rightarrow$ rms $\propto A_{eff} \Rightarrow$ SNR *independent* of A_{eff}

Quantum noise and the 2 slit paradox



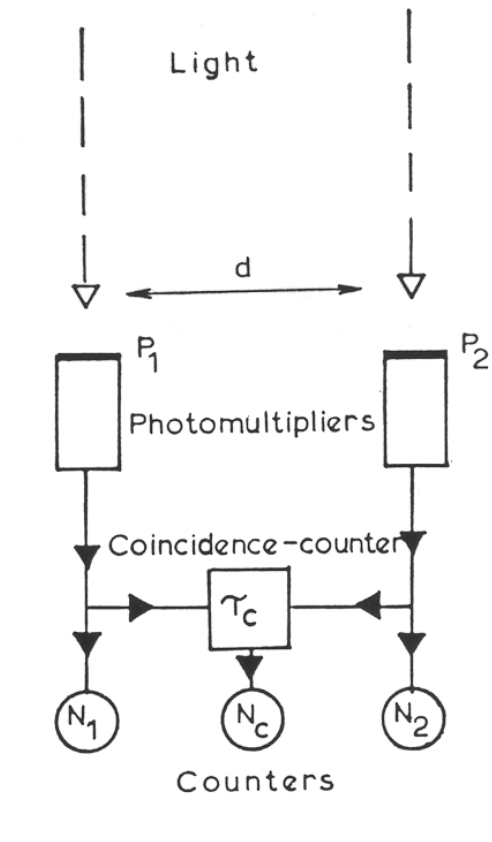


Which slit does the photon enter? With a phase conserving amplifier it seems one could both detect the photon and ‘build-up’ the interference pattern (which we know can’t be correct). But quantum noise dictates that the amplifier introduces 1 photon/mode noise, such that:

$$I_{tot} = 1 +/- 1$$

and we still cannot tell which slit the photon came through!

Intensity Interferometry: rectifying signal with square-law detector (‘photon counter’) destroys phase information. Cross correlation of intensities still results in a finite correlation, proportional to the square of E-field correlation coefficient as measured by a ‘normal’ interferometer. **Exact same phenomenon as increased correlation for t < 1/Δν in lag-space above,** ie. correlation of the wave noise itself = ‘Brown and Twiss effect’



$$\overline{N}_c = \overline{N}_1 \overline{N}_2 2\tau \left[1 + \frac{1}{2}\gamma^2 \right]$$ γ = correlation coefficient

- **Voltages correlate on timescales ~ 1/ν, with correlation coef, γ**
- **Intensities correlate on timescales ~ 1/Δν, with correlation coef, γ^2**

Advantage: timescale = 1/Δν (not 1/ν)

=> insensitive to poor optics, ‘seeing’

Disadvantage: No visibility phase information

lower SNR

Interferometric Radiometer Equation

$$S_{rms} = \frac{2kT_{sys}}{A_{eff}\sqrt{N_A(N_A - 1)t_{\mathrm{int}}\Delta\nu}}$$

- T_{sys} = wave noise for photons (RJ): rms $\propto$ total power
- A_{eff},k_B = Johnson-Nyquist noise + antenna temp definition
- $t\Delta\nu$ = # independent measurements of T_A/T_{sys} per pair of antennas
- N_A = # indep. meas. for array, or can be folded into A_{eff}