Wide-Field Imaging I: Non-Coplanar Visibilities

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Review: Measurement Equation

- From the first lecture, we have a general relation between the complex visibility V(u,v,w), and the sky intensity *I*(*l*,*m*):

$$V(u,v,w) = \iint \frac{I(l,m)}{\sqrt{1-l^2-m^2}} e^{-i2\pi[ul+vm+w(n-1)]} \, dl dm$$

- This equation is valid for:
  - spatially incoherent radiation from the far field,
  - phase-tracking interferometer
  - narrow bandwidth: $\Delta \upsilon << \frac{\theta_{res}}{\theta_{offset}} \upsilon_0 \approx \frac{\lambda}{B} \frac{D}{\lambda} \nu_0 = \frac{D}{B} \nu_0$
  - short averaging time: $\Delta t << \frac{\lambda}{B \omega_e \theta_{offset}} \approx \frac{D}{B} \frac{1}{\omega_e}$

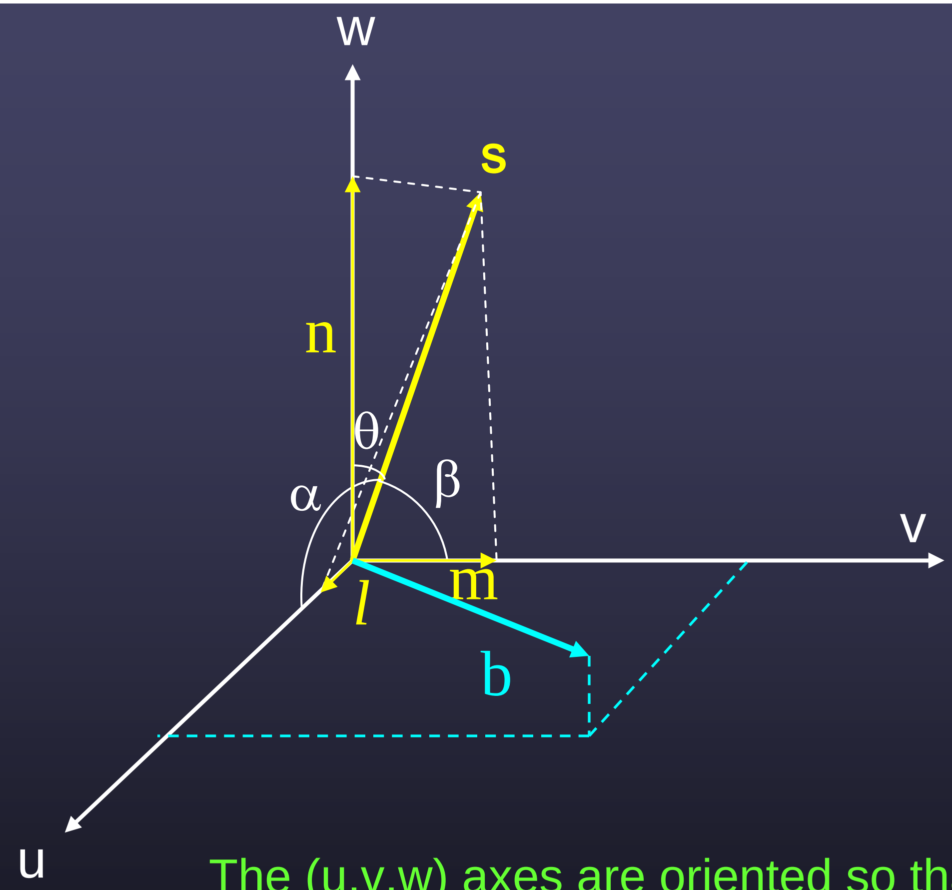
Review: Coordinate Frame

The unit direction vector **s** is defined by its projections on the (u,v,w) axes. These components are called the **Direction Cosines**, (*l,m,n*)

$$l = \cos(\alpha)$$

$$m = \cos(\beta)$$

$$n = \cos(\theta) = \sqrt{1 - l^2 - m^2}$$



The baseline vector b is specified by its coordinates (u,v,w) (measured in wavelengths).

$$\mathbf{b} = (\lambda u, \lambda v, \lambda w)$$

The (u,v,w) axes are oriented so that:

w points to the source center

u points to the East

v points to the North

When approximations fail us …

- Under certain conditions, this integral relation can be reduced to a 2-dimensional Fourier transform.
- This occurs when one of two conditions is met:
  1. All the measures of the visibility are taken on a plane, or
  2. The field of view is 'sufficiently small', given by:

$$\theta_{2D} < \sqrt{\frac{1}{\mathrm{w}}} \le \sqrt{\frac{\lambda}{\mathrm{B}}} \sim \sqrt{\theta_{syn}}$$

Worst Case!

- We are in trouble when the 'distortion-free' solid angle is smaller than the antenna primary beam solid angle.
- Define a ratio of these solid angles:

$$N_{2D} = \frac{\Omega_{PB}}{\Omega_{2D}} \sim \frac{\Omega_{PB}}{\theta_{syn}} \sim \frac{\lambda B}{D^2}$$

When $N_{2D} > 1$, 2-dimensional imaging is in trouble.

θ_{2D} and θ_{PB} for the VLA …

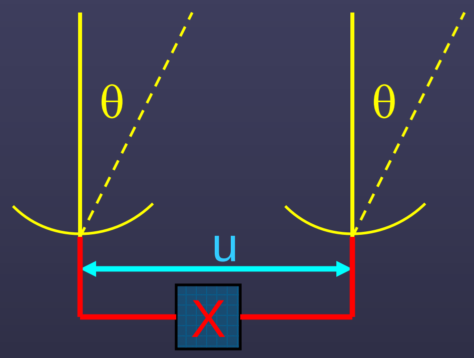
- The table below shows the approximate situation for the VLA, when it is used to image its entire primary beam.
- Blue numbers show the primary beam FWHM
- Green numbers show situations where the 2-D approximation is safe.
- Red numbers show where the approximation fails totally.

Table showing the VLA's distortion free imaging range (green), marginal zone (yellow), and danger zone (red)

λ	θ_{FWHM}	A	B	C	D
6 cm	9'	6'	10'	17'	31'
20 cm	30'	10'	18'	32'	56'
90 cm	135'	21'	37'	66'	118'
400 cm	600'	45'	80'	142'	253'

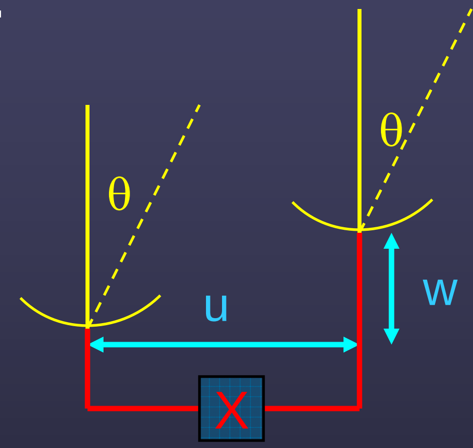
Origin of the Problem is Geometry!

- Consider two interferometers, with the same separation in ‘u’: One level, the other ‘on a hill’.



$$l = \sin\theta$$

$$n = \cos\theta$$



- What is the phase of the visibility from angle θ, relative to the vertical?
- For the level interferometer, $\phi = 2\pi ul$
- For the ‘tilted’ interferometer, $\phi = 2\pi[ul + w(n-1)]$
- These are not the same (except when θ = 0) – there is an additional phase: $\delta\phi = w(n-1)$ which is dependent both upon w and θ.
- The correct (2-d) phase is that of the level interferometer.

So – What To Do?

- If your source, or your field of view, is larger than the 'distortion-free' imaging diameter, then the 2-d approximation employed in routine imaging is not valid, and you will get a distorted image.
- In this case, we must return to the general integral relation between the image intensity and the measured visibilities.

$$V(u,v,w) = \iint \frac{I(l,m)}{\sqrt{1-l^2-m^2}} e^{-i2\pi[ul+vm+w(n-1)]}\, dldm$$

- This general relationship is not a Fourier transform. It thus doesn't have an immediate inversion to the (2-d) brightness.
- But, we can consider the 3-D Fourier transform of V(u,v,w), giving a 3-D 'image volume' F(*l*,*m*,*n*), and try relate this to the desired intensity, I(*l*,*m*).
- The mathematical details are straightforward, but tedious, and are given in detail on pp 384-385 in the White Book.

The 3-D Image Volume F(l,m,n)

- So we evaluate the following:

$$F(l,m,n) = \iiint \mathrm{V}_0(\mathrm{u},\mathrm{v},\mathrm{w}) \exp[2\pi i(\mathrm{u}l + \mathrm{v}m + \mathrm{w}n)] dudvdw$$

where

$$\mathrm{V}_0(\mathrm{u},\mathrm{v},\mathrm{w}) = \exp(-2\pi i\mathrm{w}) \mathrm{V}(\mathrm{u},\mathrm{v},\mathrm{w})$$

and try relate the function F(l,m,n) to I (l,m).

- The modified visibility $V_0(u,v,w)$ is the observed visibility with no phase compensation for the delay distance, w.

- It is the visibility, referenced to the vertical direction.

Interpretation

- *F(l,m,n)* is related to the desired intensity, I(*l,m*), by:

$$F(l,m,n) = \frac{I(l,m)}{\sqrt{1-l^2-m^2}}\,\delta\left(l^2+m^2+n^2-1\right)$$

- This states that the image volume is everywhere empty (F(*l,m,n*)=0), except on a spherical surface of unit radius where

$$l^2+m^2+n^2=1$$

- The correct sky image, I(*l,m*)/n, is the value of F(*l,m,n*) on this unit surface

- **Note: The image volume is not a physical space. It is a mathematical construct.**

Coordinates

- Where on the unit sphere are sources found?

$$l = \cos\delta \sin\Delta\alpha$$

$$m = \sin\delta \cos\delta_0 - \cos\delta \sin\delta_0 \cos\Delta\alpha$$

$$n = \sin\delta \sin\delta_0 + \cos\delta \cos\delta_0 \cos\Delta\alpha$$

where δ_0 = the reference declination, and
$\Delta\alpha$ = the offset from the reference right ascension.

However, where the sources appear on a 2-d plane is a different matter.

Benefits of a 3-D Fourier Relation

- The identification of a 3-D Fourier relation means that all the relationships and theorems mentioned for 2-d imaging in earlier lectures carry over directly.

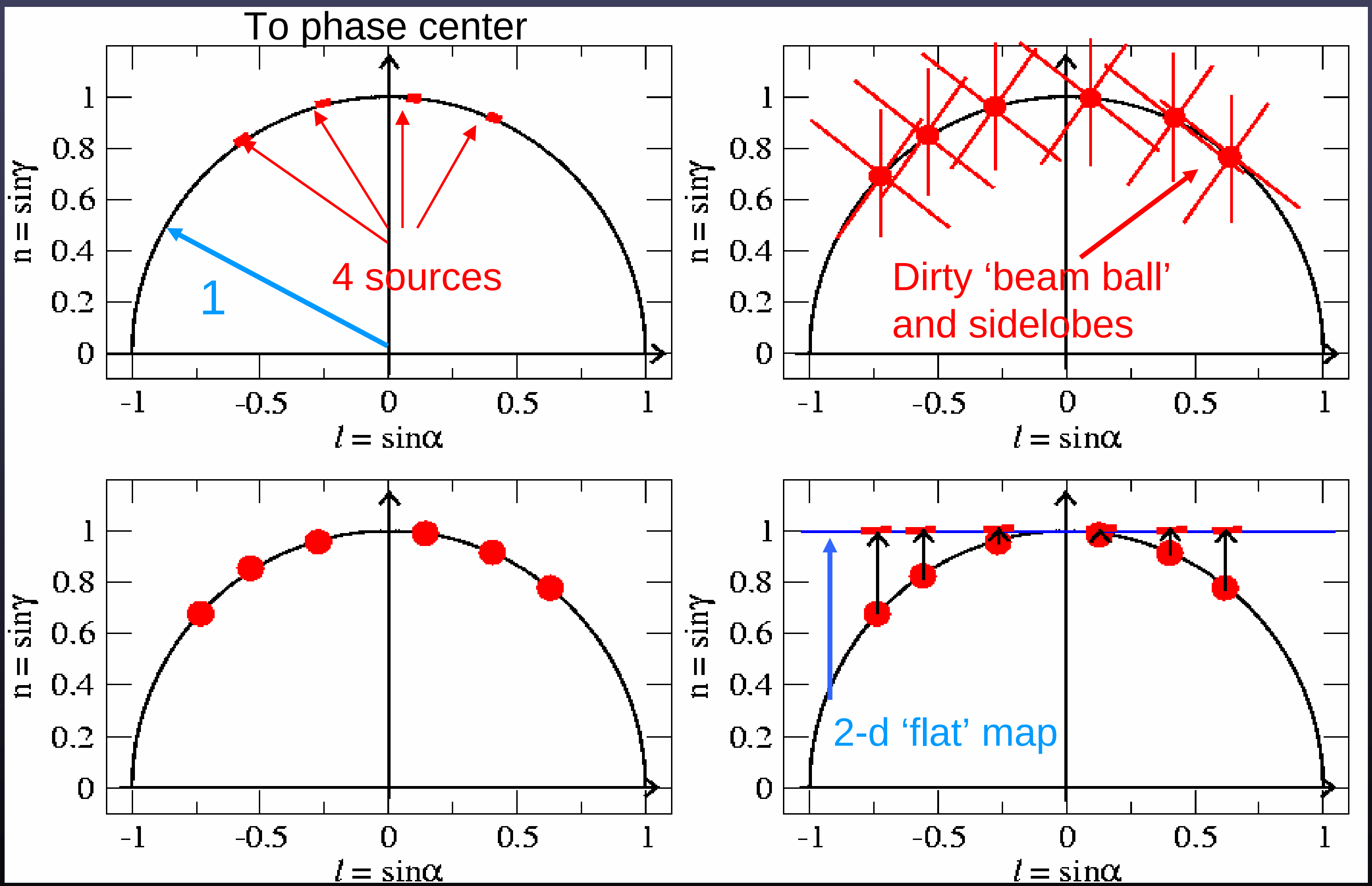
- These include:
  - Effects of finite sampling of V(u,v,w).
  - Effects of maximum and minimum baselines.
  - The ‘dirty beam’ (now a ‘beam ball’), sidelobes, etc.
  - Deconvolution, ‘clean beams’, self-calibration.

- All these are, in principle, carried over unchanged, with the addition of the third dimension.

- But the real world makes this straightforward approach unattractive (but not impossible).

Illustrative Example – a slice through the m = 0 plane

Upper Left: True Image. Upper right: Dirty Image.
Lower Left: After deconvolution. Lower right: After projection

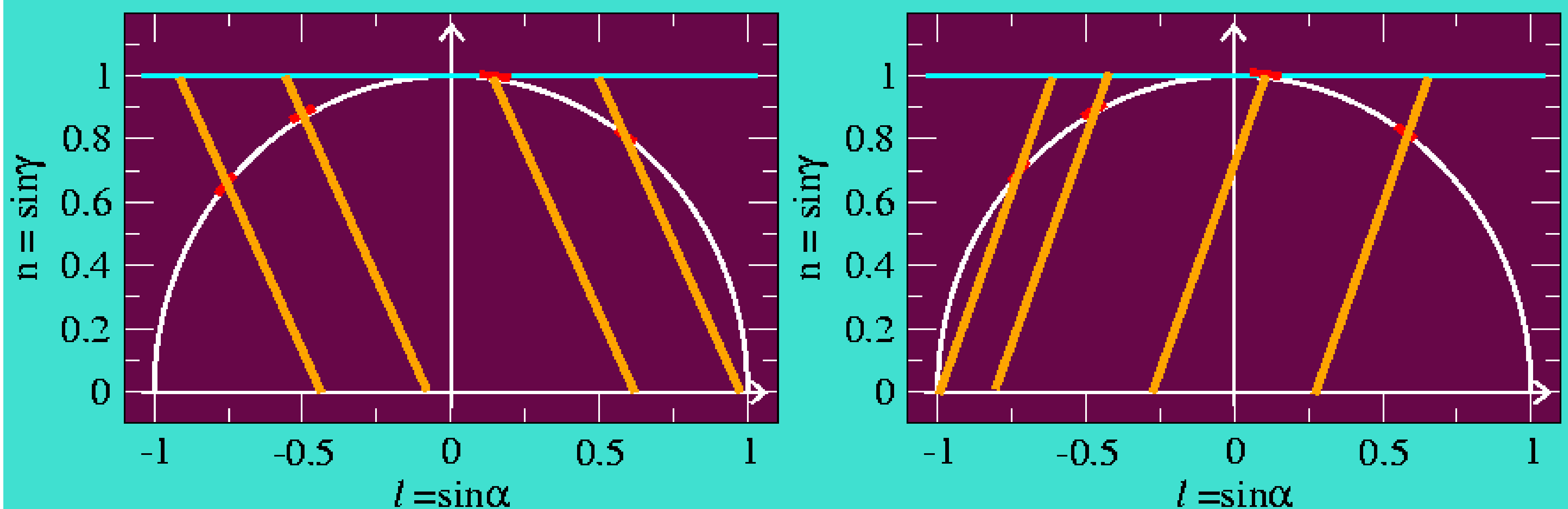


Beam Balls and Beam Rays

- In traditional 2-d imaging, the incomplete coverage of the (u,v) plane leads to rather poor “dirty beams’, with high sidelobes, and other undesirable characteristics.
- In 3-d imaging, the same number of visibilities are now distributed through a 3-d cube.
- The 3-d ‘beam ball’ is a very, very ‘dirty’ beam.
- The only thing that saves us is that the sky emission is constrained to lie on the unit sphere.
- Now consider a short observation from a coplanar array (like the VLA).
- As the visibilities lie on a plane, the instantaneous dirty beam becomes a ‘beam ray’, along an angle defined by the orientation of the plane.

Snapshots in 3D Imaging

- A deeper understanding will come from considering ‘snapshot’ observations with a coplanar array, like the VLA.
- A snapshot VLA observation, seen in ‘3D’, creates ‘beam rays’ (orange lines) , which uniquely project the sources (red bars) to the tangent image plane (blue).
- The apparent locations of the sources on the 2-d tangent map plane move in time, except for the tangent position (phase center).



Apparent Source Movement

- As seen from the sky, the plane containing the VLA changes its tilt through the day.
- This causes the 'beam rays' associated with the snapshot images to rotate.
- The apparent source position in a 2-D image thus moves, following a conic section. The locus of the path (l',m') is:

$$l' = l - \left(1 - \sqrt{1 - l^2 - m^2}\right) \tan Z \sin \Psi_P$$
$$m' = m + \left(1 - \sqrt{1 - l^2 - m^2}\right) \tan Z \cos \Psi_P$$

where Z = the zenith distance, Ψ_P = parallactic angle, and (l,m) are the correct coordinates of the source.

Wandering Sources

- The apparent source motion is a function of zenith distance and parallactic angle, given by:

$$\tan\chi = \frac{\cos\phi \sin H}{\sin\phi \cos\delta - \cos\phi \sin\delta \cos H}$$

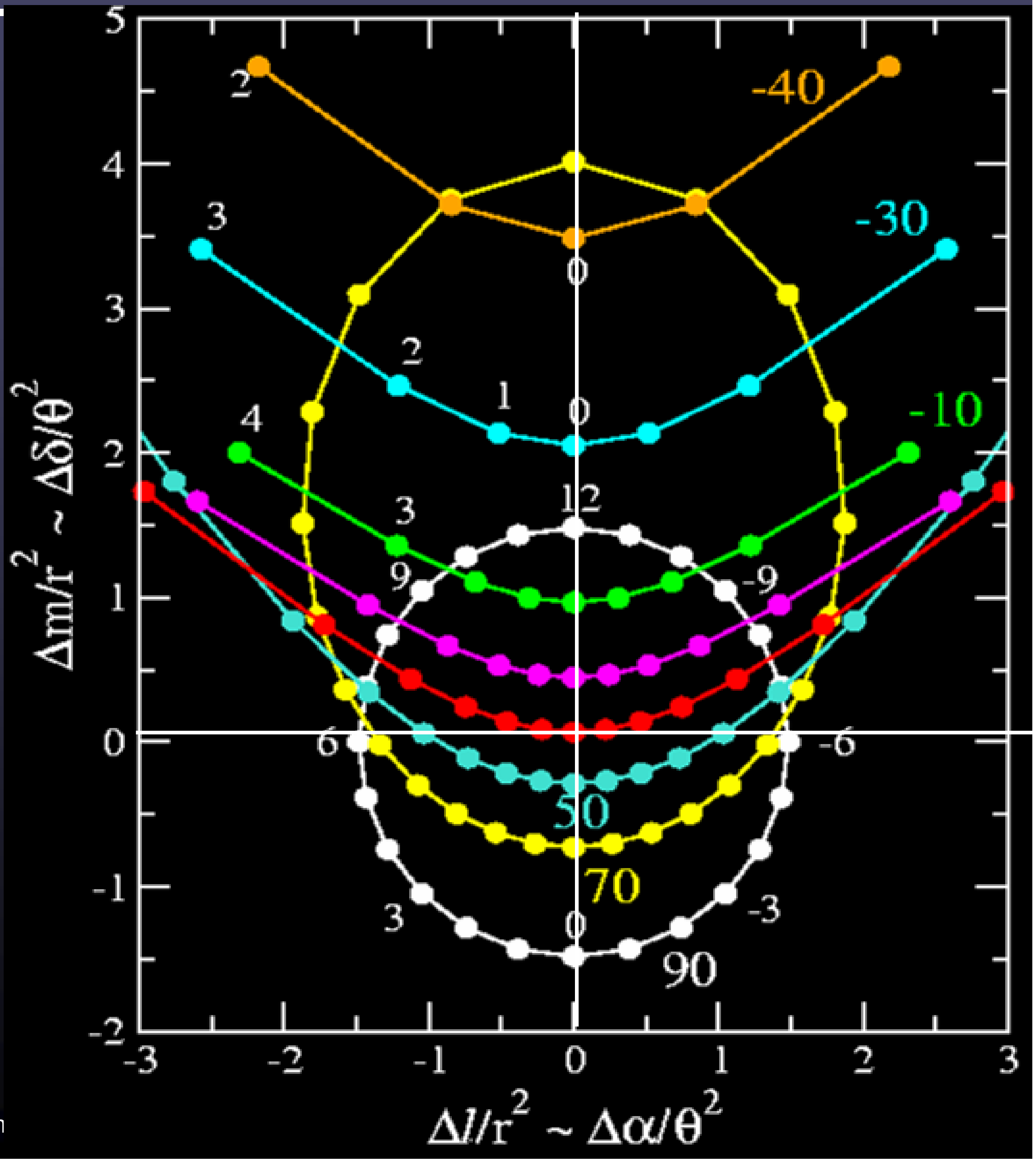
$$\cos Z = \sin\phi \sin\delta + \cos\phi \cos\delta \cos H$$

where

H = hour angle
δ = declination
ϕ = array latitude

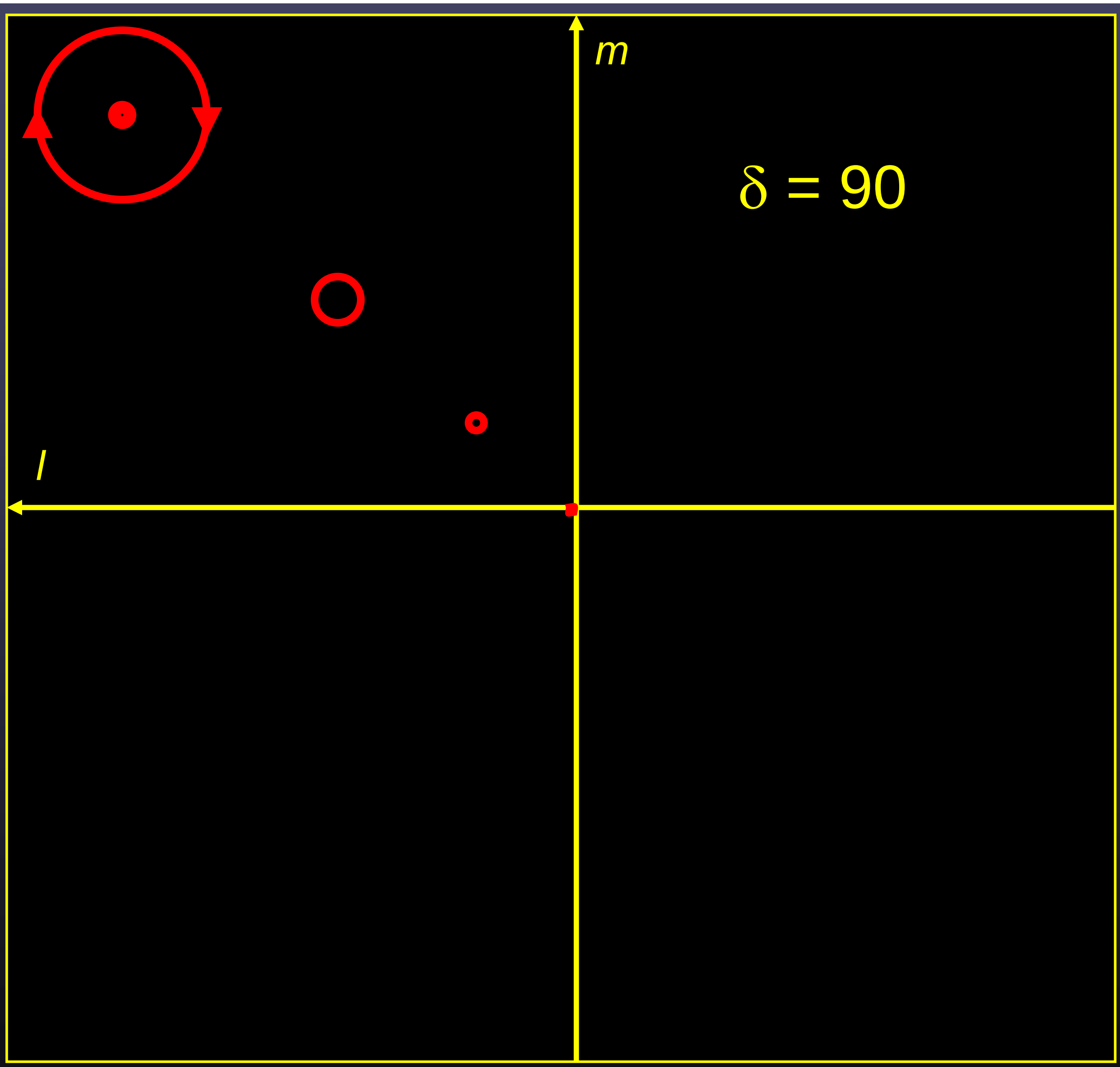
Examples of the source loci for the VLA

- On the 2-d (tangent) image plane, source positions follow conic sections.
- The plots show the loci for declinations 90, 70, 50, 30, 10, -10, -30, and -40.
- Each dot represents the location at integer HA.
- The path is a circle at declination 90.
- The only observation with no error is at HA=0, δ=34.
- The offset position scales quadratically with source offset from the phase center.



Schematic Example

- Imagine a 24-hour observation of the north pole. The `simple' 2-d output map will look something like this.
- The red circles represent the apparent source structures.
- Each doubling of distance from the phase center quadruples the extent of the distorted image.



How bad is it?

- The offset is $(1 - \cos\theta)\tan Z \sim (\theta^2 \tan Z)/2$ radians
- For a source at the antenna beam first null, $\theta \sim \lambda/D$
- So the offset, ε, measured in synthesized beamwidths, (λ/B) at the first zero of the antenna beam can be written as

$$\varepsilon = \frac{\lambda B}{2D^2}\tan Z$$

B = maximum baseline
D = antenna diameter
Z = zenith distance
λ = wavelength

- For the VLA's A-configuration, this offset error, at the antenna beam half-maximum, can be written:

$\varepsilon \sim \lambda_{cm}$ (tan Z)/20 (in beamwidths)

- This is very significant at meter wavelengths, and at high zenith angles (low elevations).

So, What Can We Do?

- There are a number of ways to deal with this problem.

1. Compute the entire 3-d image volume via FFT.
   - The most straightforward approach, but hugely wasteful in computing resources!
   - The minimum number of 'vertical planes' needed is:

     $N_{2D} \sim B\theta^2/\lambda \sim \lambda B/D^2$
   - The number of volume pixels to be calculated is: $N_{pix} \sim 4B^3\theta^4/\lambda^3 \sim 4\lambda B^3/D^4$
   - But the number of pixels actually needed is: $4B^2/D^2$
   - So the fraction of the pixels in the final output map actually used is: $D^2/\lambda B$. (~ 2% at $\lambda = 1$ meter in A-configuration!)
   - But – at higher frequencies, ($\lambda < 6$cm?), this approach might be feasible.

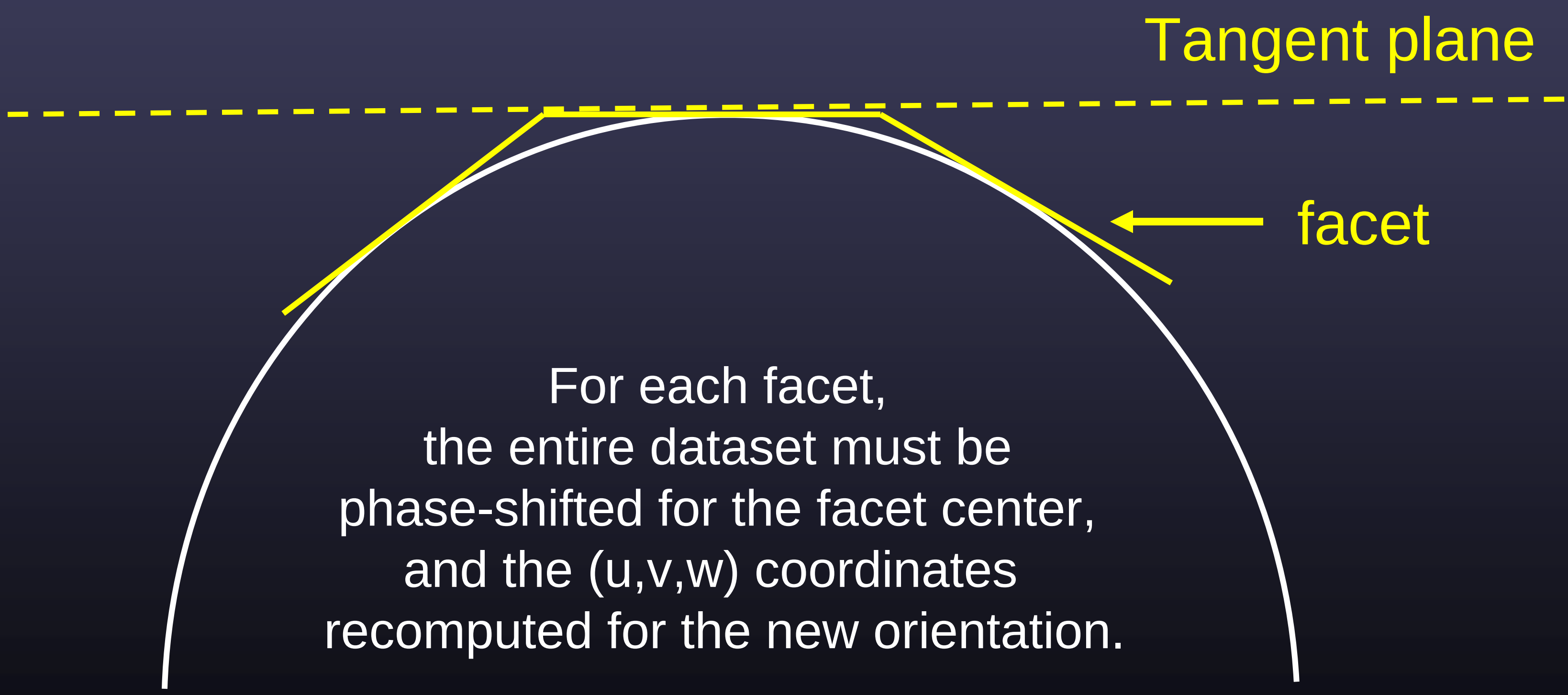
Deep Cubes!

- To give an idea of the scale of processing, the table below shows the number of ‘vertical’ planes needed to encompass the VLA’s primary beam.
- For the A-configuration, each plane is at least 2048 x 2048.
- For the New Mexico Array, it’s at least 16384 x 16384!
- And one cube would be needed for each spectral channel, for each polarization!

λ	NMA	A	B	C	D	E
400cm	2250	225	68	23	7	2
90cm	560	56	17	6	2	1
20cm	110	11	4	2	1	1
6cm	40	4	2	1	1	1
2cm	10	2	1	1	1	1
1.3cm	6	1	1	1	1	1

2. Polyhedron Imaging

- In this approach, we approximate the unit sphere with small flat planes (‘facets’), each of which stays close to the sphere’s surface.



Polyhedron Approach, (cont.)

- How many facets are needed?
- If we want to minimize distortions, the plane mustn't depart from the unit sphere by more than the synthesized beam, λ/B. Simple analysis (see the book) shows the number of facets will be:

$$N_f \sim 2\lambda B/D^2$$

or twice the number of planes needed for 3-D imaging.

- But the size of each image is much smaller, so the total number of cells computed is much smaller.
- The extra effort in phase shifting and (u,v,w) rotation is more than made up by the reduction in the number of cells computed.
- This approach is the current standard in AIPS.

Polyhedron Imaging

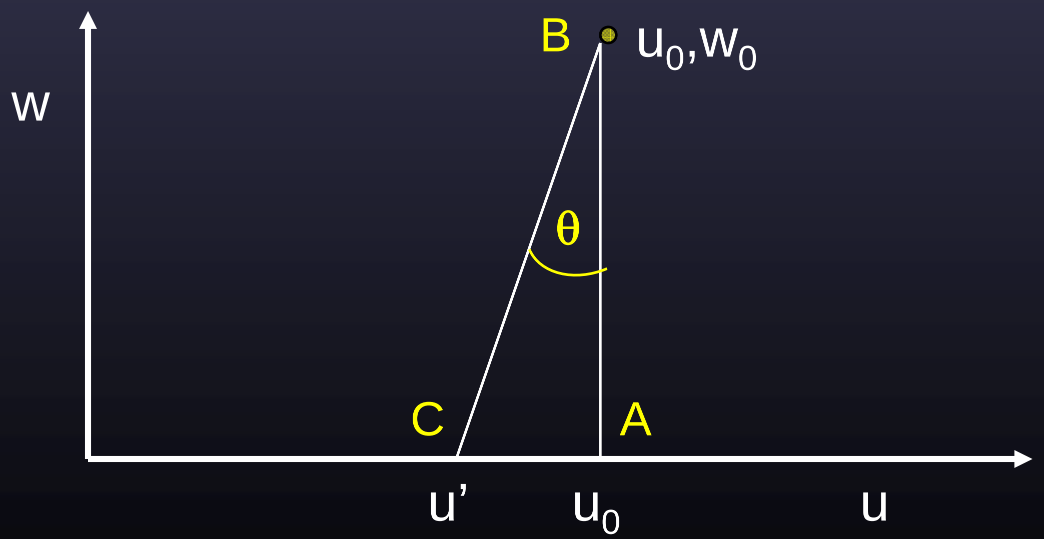
- Procedure is then:
  - Determine number of facets, and the size of each.
  - Generate each facet image, rotating the (u,v,w) and phase-shifting the phase center for each.
  - Jointly deconvolve all facets. The Clark/Cotton/Schwab major/minor cycle system is well suited for this.
  - Project the finished images onto a 2-d surface.
- Added benefit of this approach:
  - As each facet is independently generated, one can imagine a separate antenna-based calibration for each.
  - Useful if calibration is a function of direction as well as time.
  - This is needed for meter-wavelength imaging at high resolution.

W-Projection

- Although the polyhedron approach works well, it is expensive, as all the data have to be phase shifted, rotated, and gridded for each facet, and there are annoying boundary issues – where the facets overlap.
- Is it possible to reduce the observed 3-d distribution to 2-d, through an appropriate projection algorithm?
- Fundamentally, the answer appears to be NO, unless you know, in advance, the brightness distribution over the sky.
- But, it appears an accurate approximation can be done, through an algorithm originated by Tim Cornwell.
- This algorithm permits a single 2-d image and deconvolution, and eliminates the annoying edge effects which accompany the faceting approach.

W-Projection Basics

- Consider three visibilities, measured at A, B, and C, for a source.
- At A = $(u_0,0)$, for a given direction, $\phi = 2\pi u_0 \sin\theta$
- At B = (u_0,w_0), $\phi = 2\pi(u_0 \sin\theta + w_0 \cos\theta)$
- At C = $(u' = u_0 - w_0 \tan\theta, 0)$, $\phi = 2\pi(u_0 \sin\theta - w_0 \sin\theta \tan\theta)$
- The visibility at B due to a source at a given direction $l = \sin\theta$ can be converted to the correct value at A or C simply by adjusting the phase by $\delta\phi = 2\pi x$, where $x = w_0/\cos\theta$ is the propagation distance.
- Visibilities propagate the same way as an EM wave!



W-Projection

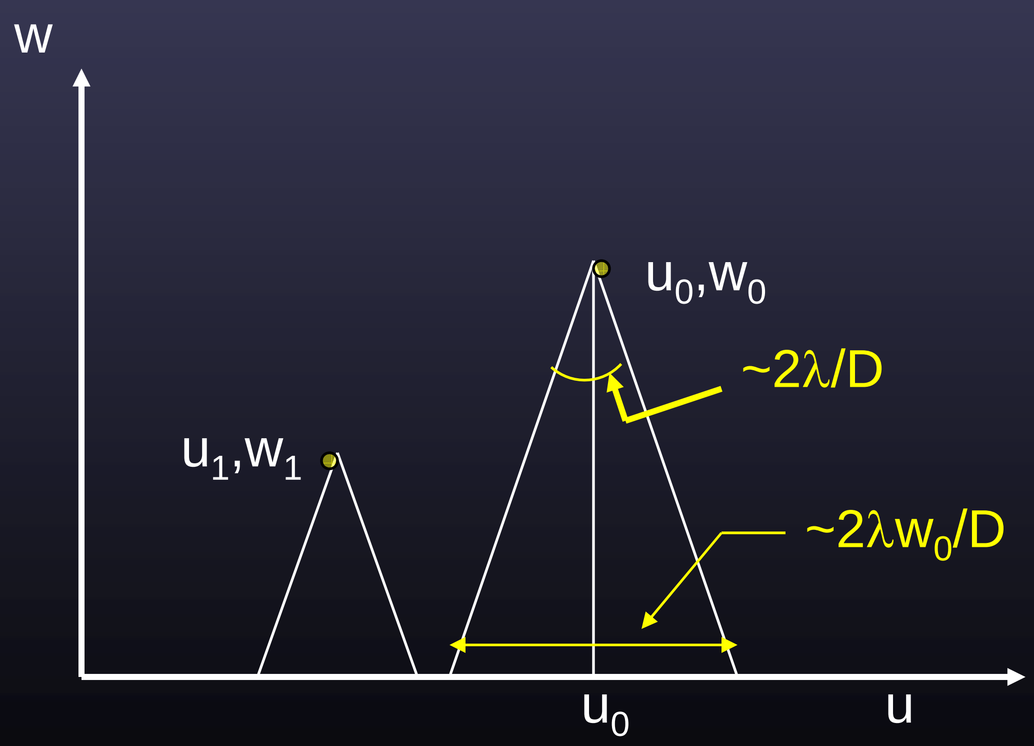
- However – to correctly project each visibility onto the plane, you need to know, in advance, the sky brightness distribution, since the measured visibility is a complex sum of visibilities from all sources:

$$V = \sum_j A_j e^{-i2\pi(ul_j + vm_j + wn_j)}$$

- Each component of this net vector must be independently projected onto its appropriate new position, with a phase adjustment given by the distance to the plane.
- In fact, standard 2-d imaging utilizes this projection – but all visibilities are projected by the vertical distance, w.
- If we don't know the brightness in advance, we can still project the visibilities over all the cells within the field of view of interest, using the projection phase (Fresnel diffraction phase).
- The maximum field of view is that limited by the antenna primary beam, $\theta \sim \lambda/D$

W-Projection

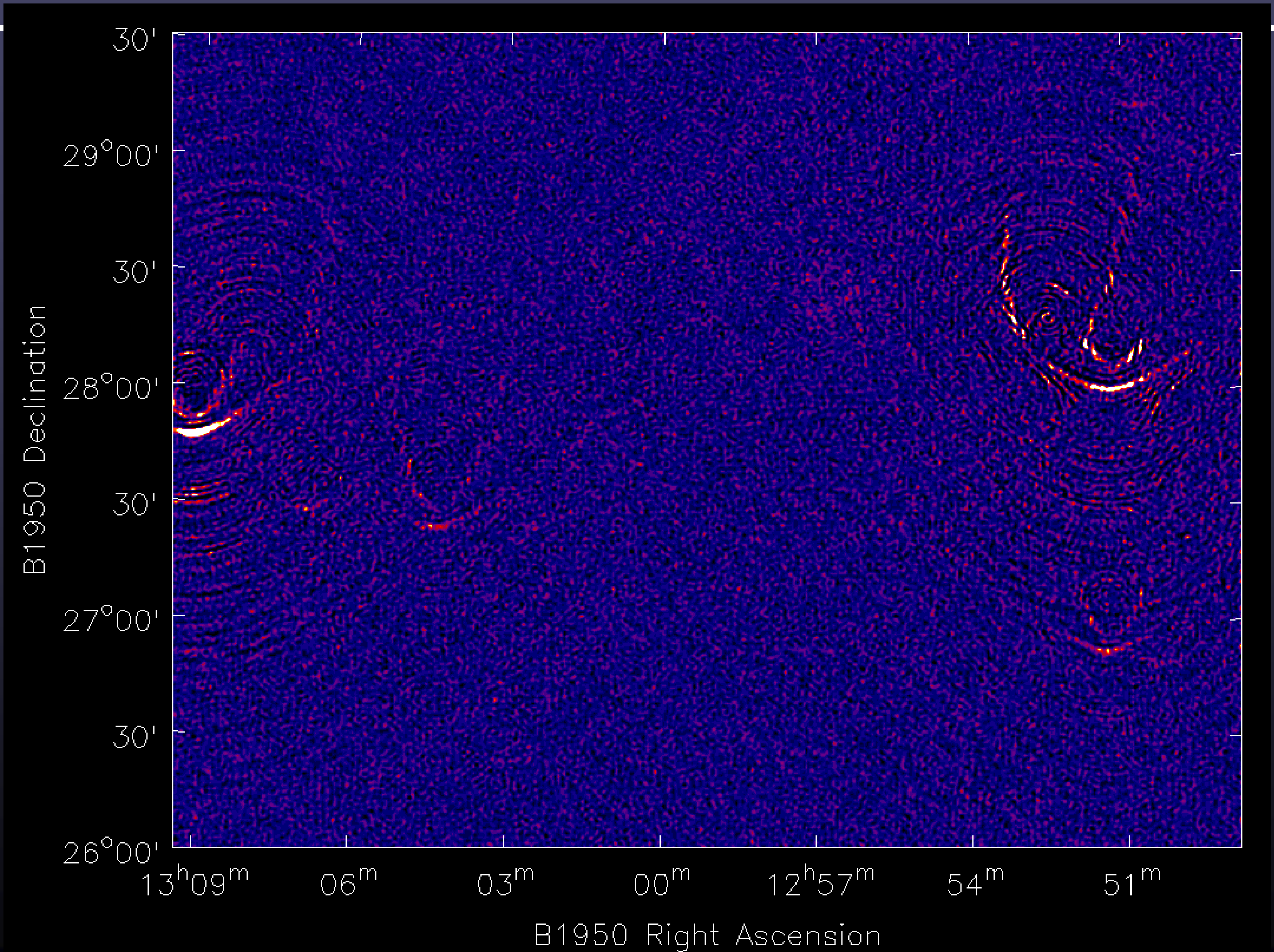
- Each visibility, at location (u,v,w) is mapped to the w=0 plane, with a phase shift proportional to the distance from the point to the plane.
- Each visibility is mapped to ALL the points lying within a cone whose full angle is the same as the field of view of the desired map – $\sim 2\lambda/D$ for a full-field image.
- Clearly, processing is minimized by minimizing w: Don't observe at large zenith angles!!!



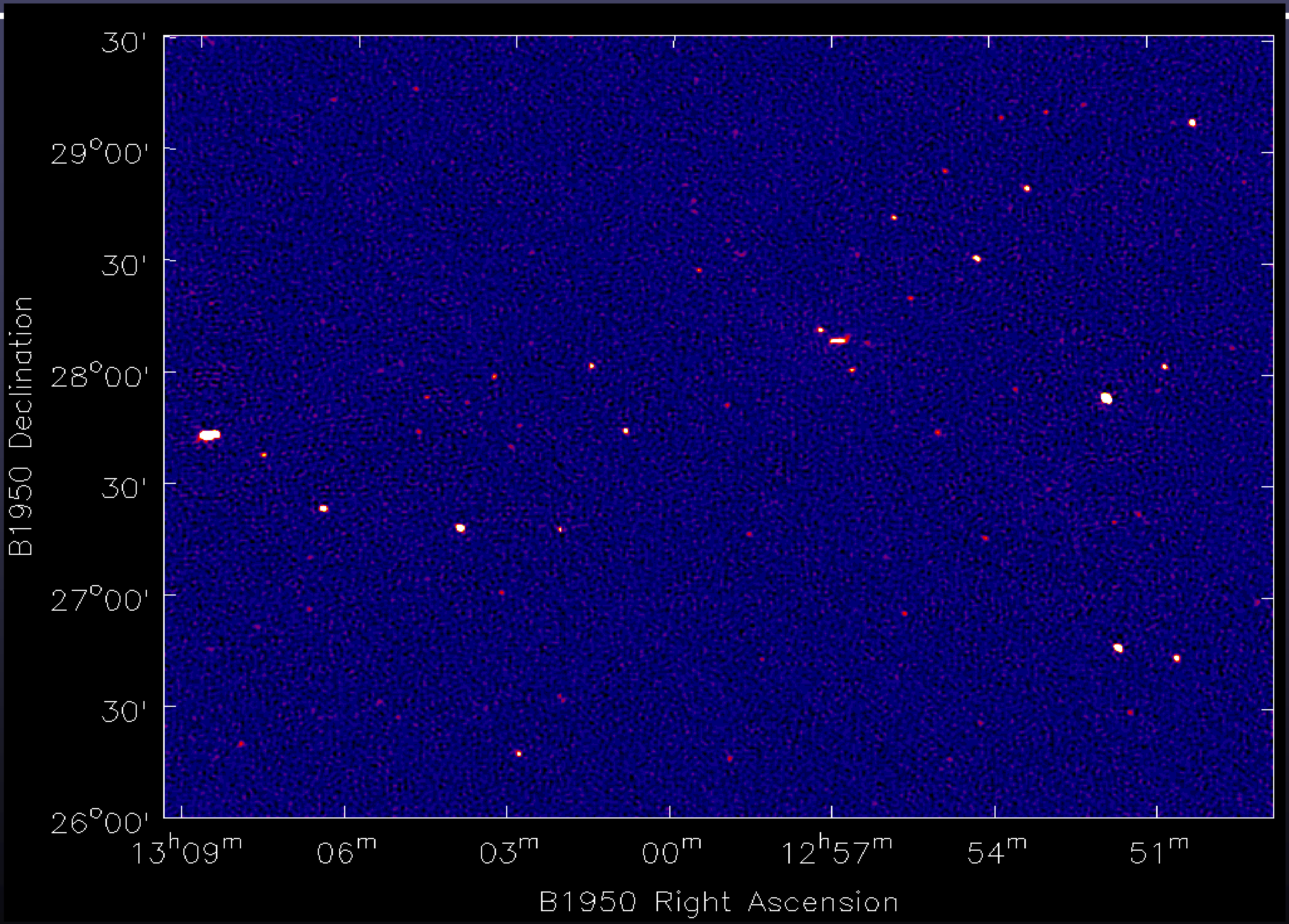
Where can W-Projection be found?

- The W-Projection algorithm is not (yet?) available in AIPS, but is available in CASA.
- The CASA version is a trial one – it needs more testing on real data.
- The authors (Cornwell, Kumar, Bhatnagar) have shown that 'W-Projection' is often very much faster than the facet algorithm – by over an order of magnitude in most cases.
- W-Projection can also incorporate spatially-variant antenna-based phase errors – include these in the phase projection for each measured visibility.
- Trials done so far give very impressive results.

An Example – without '3-D' Procesesing



Example – with 3D processing



Conclusion (of sorts)

- Arrays which measure visibilities within a 3-dimensional (u,v,w) volume, such as the VLA, cannot use a 2-d FFT for wide-field and/or low-frequency imaging.
- The distortions in 2-d imaging are large, growing quadratically with distance, and linearly with wavelength.
- In general, a 3-d imaging methodology is necessary.
- Recent research shows a Fresnel-diffraction projection method is the most efficient, although the older polyhedron method is better known.
- Undoubtedly, better ways can yet be found.