Wide Field Imaging I: Non-Coplanar Arrays

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Introduction

- From the first lecture, we have a general relation between the complex visibility V(u,v,w), and the sky intensity *I*(*l*,*m*):

$$V(u,v,w) = \iint I(l,m)\exp\{-2\pi i[ul + vm + w(n-1)]\}\, dl dm / n$$

where $n = \sqrt{1 - l^2 - m^2}$

- This equation is valid for:
  - spatially incoherent radiation from the far field,
  - phase-tracking interferometer
  - narrow bandwidth
- What is ‘narrow bandwidth’?
- $$\Delta\upsilon << \frac{\theta_{res}}{\theta_{offset}}\upsilon_0 = \frac{\lambda}{B}\frac{D}{\lambda}\upsilon_0 = \frac{D}{B}\upsilon_0$$

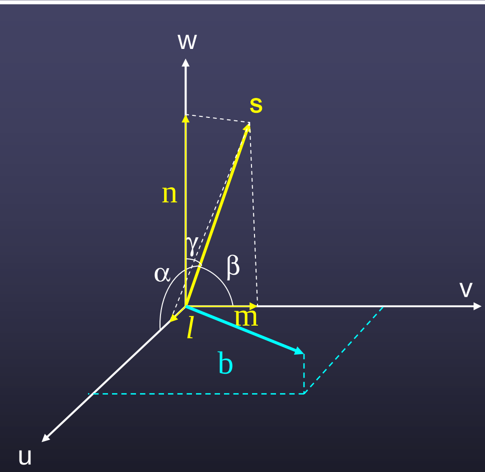
Review: Coordinate Frame

The unit direction vector **s** is defined by its projections on the (u,v,w) axes. These components are called the **Direction Cosines**, (*l*,*m*,*n*)

$$l = \cos(\alpha)$$

$$m = \cos(\beta)$$

$$n = \cos(\gamma) = \sqrt{1 - l^2 - m^2}$$



The baseline vector b is specified by its coordinates (u,v,w) (measured in wavelengths).

$$\mathbf{b} = (\lambda u, \lambda v, \lambda w)$$

When approximations fail us …

- Under certain conditions, this integral relation can be reduced to a 2-dimensional Fourier transform.
- This occurs when one of two conditions are met:
  1. All the measures of the visibility are taken on a plane, or
  2. The field of view is 'sufficiently small', given by:

$$\theta_{\max} < \sqrt{\frac{1}{\mathrm{w}}} \le \sqrt{\frac{\lambda}{\mathrm{B}}} \sim \sqrt{\theta_{syn}}$$

Table showing the VLA's distortion free imaging range (green), marginal zone (yellow), and danger zone (red)

λ	θ_{ant}	A	B	C	D
6 cm	9'	6'	10'	17'	31'
20 cm	30'	10'	18'	32'	56'
90 cm	135'	21'	37'	66'	118'
400 cm	600'	45'	80'	142'	253'

Not a 3-D F.T. – but let's do it anyway …

- If your source, or your field of view, is larger than the 'distortion-free' imaging diameter, then the 2-d approximation employed in routine imagine are not valid, and you will get a crappy image.
- In this case, we must return to the general integral relation between the image intensity and the measured visibilities.
- The general relationship is not a Fourier transform. It thus doesn't have an immediate inversion.
- But, we can consider the 3-D Fourier transform of V(u,v,w), giving a 3-D 'image volume' F(*l*,*m*,*n*), and try relate this to the desired intensity, I(*l*,*m*).
- The mathematical details are straightforward, but tedious, and are given in detail on pp 384-385 in the White Book.

The 3-D Image Volume

- We find that:

$$F(l,m,n) = \iiint \mathrm{V_0(u,v,w)} \exp[2\pi i(\mathrm{u}l + \mathrm{v}m + \mathrm{w}n)] d\mathrm{u} d\mathrm{v} d\mathrm{w}$$

where

$$\mathrm{V_0(u,v,w)} = \exp(-2\pi i \mathrm{w}) \mathrm{V(u,v,w)}$$

- *F(l,m,n)* is related to the desired intensity, I(*l*,*m*), by:

$$F(l,m,n) = \frac{I(l,m)}{\sqrt{1-l^2-m^2}} \delta\left(\sqrt{1-l^2-m^2} - 1\right)$$

This relation looks daunting, but in fact has a lovely geometric interpretation.

Interpretation

- The modified visibility $V_0(u,v,w)$ is simply the observed visibility with no 'fringe tracking'.
- It's what we would measure if the fringes were held fixed, and the sky moves through them.
- The bottom equation states that the image volume is everywhere empty ($F(l,m,n)=0$), except on a spherical surface of unit radius where

$$l^2 + m^2 + n^2 = 1$$

- The correct sky image, $I(l,m)/n$, is the value of $F(l,m,n)$ on this unit surface
- Note: The image volume is not a physical space. It is a mathematical construct.

Benefits of a 3-D Fourier Relation

- The identification of a 3-D Fourier relation means that all the relationships and theorems mentioned for 2-d imaging in earlier lectures carry over directly.
- These include:
  - Effects of finite sampling of V(u,v,w).
  - Effects of maximum and minimum baselines.
  - The 'dirty beam' (now a 'beam ball'), sidelobes, etc.
  - Deconvolution, 'clean beams', self-calibration.
- All these are, in principle, carried over unchanged, with the addition of the third dimension.
- But the real world makes this straightforward approach unattractive (but not impossible).

Coordinates

- Where on the unit sphere are sources found?

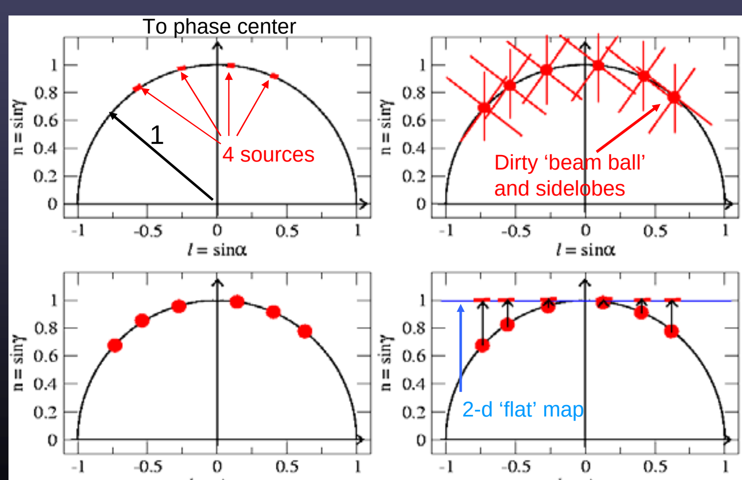
$$l = \cos\delta \sin\Delta\alpha$$
$$m = \sin\delta\cos\delta_0 - \cos\delta\sin\delta_0\cos\Delta\alpha$$
$$n = \sin\delta\sin\delta_0 + \cos\delta\cos\delta\ \cos\Delta\alpha$$

where δ_0 = the reference declination, and
$\Delta\alpha$ = the offset from the reference right ascension.

However, where the sources appear on a 2-d plane is a different matter.

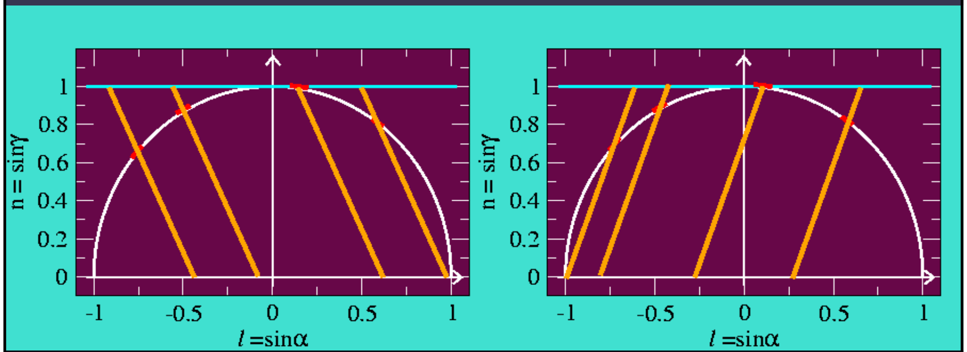
Illustrative Example – a slice through the m = 0 plane

Upper Left: True Image. Upper right: Dirty Image.
Lower Left: After deconvolution. Lower right: After projection



Snapshots in 3D Imaging

- A snapshot VLA observation, seen in '3D', creates 'line beams' (orange lines) , which uniquely project the sources (red bars) to the image plane (blue).
- Except for the tangent point, the apparent locations of the sources move in time.



Apparent Source Movement

- As seen from the sky, the plane containing the VLA rotates through the day.
- This causes the 'line-beams' associated with the snapshot images to rotate.
- The apparent source position in a 2-D image thus rotates, following a conic section. The loci of the path is:

$$l' = l - \left(1 - \sqrt{1 - l^2 - m^2}\right) \tan Z \sin\Psi_P$$

$$m' = m + \left(1 - \sqrt{1 - l^2 - m^2}\right) \tan Z \cos\Psi_P$$

where Z = the zenith distance, and Ψ_P = parallactic angle,
And (l,m) are the correct angular coordinates of the source.

Wandering Sources

- The apparent source motion is a function of zenith distance and parallactic angle, given by:

$$\tan\chi = \frac{\cos\phi\sin H}{\sin\phi\cos\delta - \cos\phi\sin\delta\cos H}$$

$$\cos Z = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H$$

where

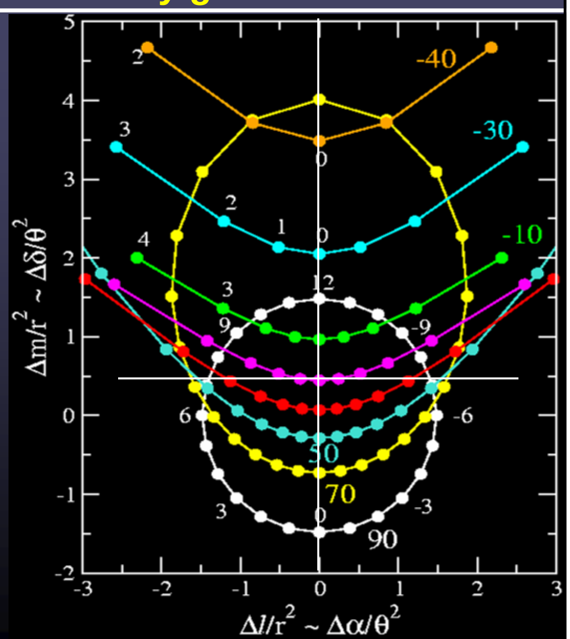
H = hour angle

δ = declination

ϕ = antenna latitude

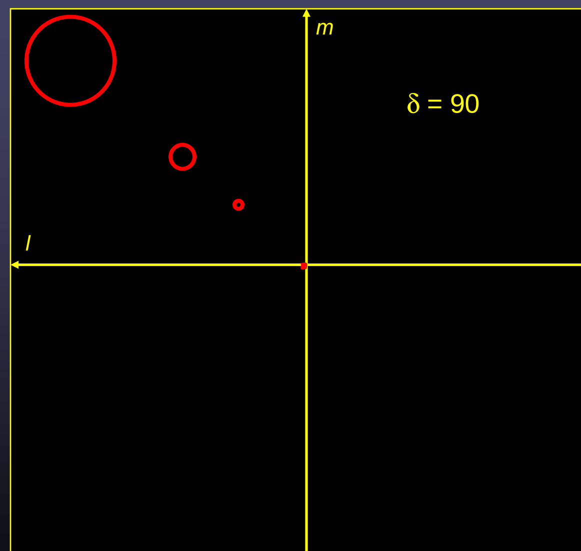
And around they go …

- On the 2-d (tangent) image plane, source positions follow conic sections.
- The plots show the loci for declinations 90, 70, 50, 30, 10, -10, -30, and -40.
- Each dot represents the location at integer HA.
- The path is a circle at declination 90.
- The only observation with no error is at HA=0, δ=34.
- The error scales quadratically with source offset from the phase center.



Schematic Example

- Imagine a 24-hour observation of the north pole. The `simple' 2-d output map will look something like this.
- The red circles represent the apparent source structures.
- Each doubling of distance from the phase center quadruples the extent of the distorted image.



How bad is it?

- In practical terms …
- The offset is (1 - cos γ) tan Z ~ (γ^2 tan Z)/2 radians
- For a source at the antenna beam half-power, $\gamma \sim \lambda/2D$
- So the offset, ε, measured in synthesized beamwidths, (λ/B) at the half-power of the antenna beam can be written as

$$\varepsilon = \frac{\lambda B}{8D^2} \tan Z$$

B = maximum baseline
D = antenna diameter
Z = zenith distance
λ = wavelength

- For the VLA's A-configuration, this offset error, at the antenna FWHM, can be written:

$\varepsilon \sim \lambda_{cm}$ (tan Z)/20 (in beamwidths)

- This is very significant at meter wavelengths, and at high zenith angles (low elevations).

So, What can we do?

- There are a number of ways to deal with this problem.

1. Compute the entire 3-d image volume.
   - The most straightforward approach, but hugely wasteful in computing resources!
   - The minimum number of 'vertical planes' needed is:
     $N_n \sim B\theta^2/\lambda \sim \lambda B/D^2$
   - The number of volume pixels to be calculated is:
     $N_{pix} \sim 4B^3\theta^4/\lambda^3 \sim 4\lambda B^3/D^4$
   - But the number of pixels actually needed is: $4B^2/D^2$
   - So the fraction of the pixels in the final output map actually used is: $D^2/\lambda B$. (~ 2% at λ = 1 meter in A-configuration!)

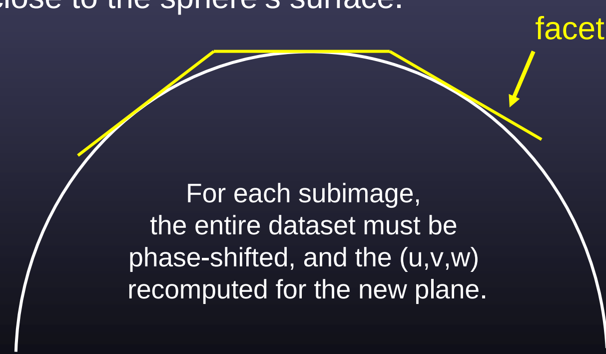
Deep Cubes!

- To give an idea of the scale of processing, the table below shows the number of 'vertical' planes needed to encompass the VLA's primary beam.
- For the A-configuration, each plane is at least 2048 x 2048.
- For the New Mexico Array, it's at least 16384 x 16384!
- And one cube would be needed for each spectral channel, for each polarization!

λ	NMA	A	B	C	D	E
400cm	2250	225	68	23	7	2
90cm	560	56	17	6	2	1
20cm	110	11	4	2	1	1
6cm	40	4	2	1	1	1
2cm	10	2	1	1	1	1
1.3cm	6	1	1	1	1	1

2. Polyhedron Imaging

- The wasted effort is in computing pixels we don't need.
- The polyhedron approach approximates the unit sphere with small flat planes, each of which stays close to the sphere's surface.



Polyhedron Approach, (cont.)

- How many facets are needed?
- If we want to minimize distortions, the plane mustn't depart from the unit sphere by more than the synthesized beam, λ/B. Simple analysis (see the book) shows the number of facets will be:

$$N_f \sim 2\lambda B/D^2$$

or twice the number needed for 3-D imaging.

- But the size of each image is much smaller, so the total number of cells computed is much smaller.
- The extra effort in phase computation and (u,v,w) rotation is more than made up by the reduction in the number of cells computed.
- This approach is the current standard in AIPS.

Polyhedron Imaging

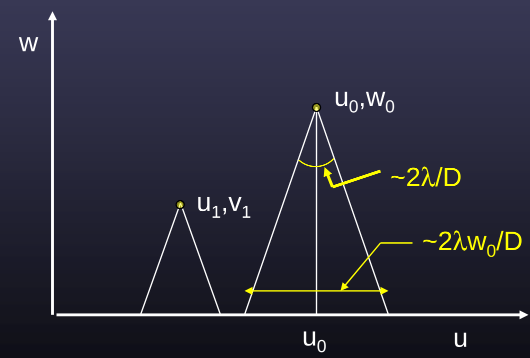
- Procedure is then:
  - Determine number of facets, and the size of each.
  - Generate each facet image, rotating the (u,v,w) and phase-shifting the phase center for each.
  - Jointly deconvolve the set. The Clark/Cotton/Schwab major/minor cycle system is well suited for this.
  - Project the finished images onto a 2-d surface.
- Added benefit of this approach:
  - As each facet is independently generated, one can imagine a separate antenna-based calibration for each.
  - Useful if calibration is a function of direction as well as time.
  - This is needed for meter-wavelength imaging.

W-Projection

- Although the polyhedron approach works well, it is expensive, and there are annoying boundary issues – where the facets overlap.
- Is it possible to project the data onto a single (u,v) plane, accounting for all the necessary phase shifts?
- Answer is YES! Tim Cornwell has developed a new algorithm, termed 'w-projection', to do this.
- Available only in CASA (formerly known as AIPS++), this approach permits a single 2-d image and deconvolution, and eliminates the annoying edge effects which accompany re-projection.

W-Projection

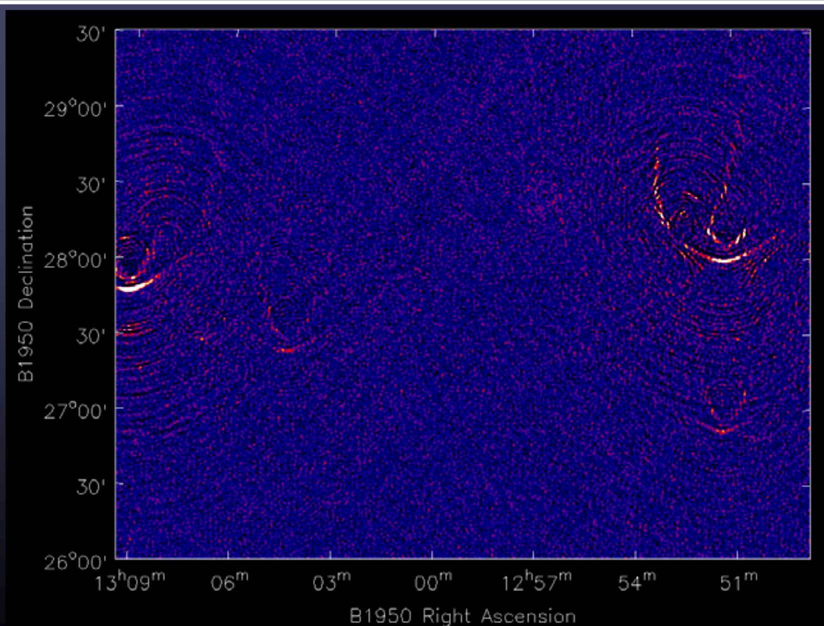
- Each visibility, at location (u,v,w) is mapped to the w=0 plane, with a phase shift proportional to the distance.
- Each visibility is mapped to ALL the points lying within a cone whose full angle is the same as the field of view of the desired map – $\sim 2\lambda/D$ for a full-field image.
- Area in the base of the cone is $\sim 4\lambda^2 w^2/D^2 < 4B^2/D^2$. Number of cells on the base which 'receive' this visibility is $\sim 4w_0^2B^2/D^2 < 4B^4/\lambda^2D^2$.



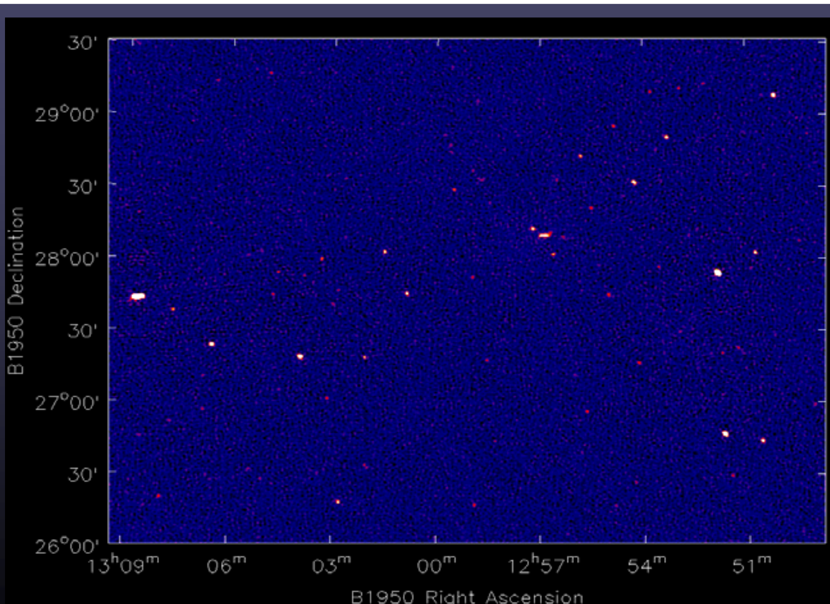
W-Projection

- The phase shift for each visibility onto the w=0 plane is in fact a Fresnel diffraction function.
- Each 2-d cell receives a value for each observed visibility within an (upward/downwards) cone of full angle $\theta < \lambda/D$ (the antenna's field of view).
- In practice, the data are non-uniformly vertically gridded – speeds up the projection.
- There are a lot of computations, but they are done only once.
- Spatially-variant self-cal can be accommodated (but hasn't yet).

An Example – without '3-D' Procesesing



Example – with 3D processing



Conclusion (of sorts)

- Arrays which measure visibilities within a 3-dimensional (u,v,w) volume, such as the VLA, cannot use a 2-d FFT for wide-field and/or low-frequency imaging.
- The distortions in 2-d imaging are large, growing quadratically with distance, and linearly with wavelength.
- In general, a 3-d imaging methodology is necessary.
- Recent research shows a Fresnel-diffraction projection method is the most efficient, although the older polyhedron method is better known.
- Undoubtedly, better ways can yet be found.