Noise in Interferometry

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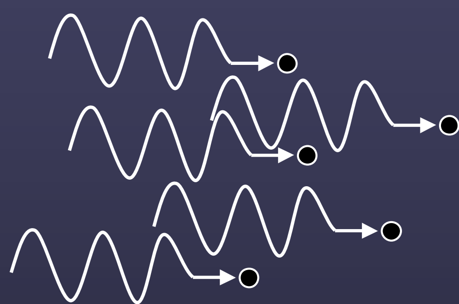
Overview

- Fundamental origins of noise
  – Signal-dependent noise
  – Detector noise
- Practical noise considerations in interferometers
  – Radio vs. optical interferometry
  – Wave vs particle noise
- Sensitivity analysis for standard image estimators
  – Radio vs optical
  – Fringe phasor vs power spectrum and bispectrum
  – Theoretical expressions/limits for the various SNR
- Concluding remarks

Electromagnetic radiation and associated noise sources

- Intrinsic quantum mechanical uncertainty
  - EM radiation consists of discrete photons ➔ associated uncertainty of random arrival times
  - Photons can possess intrinsic correlations ➔ wide range of associated noise behaviors, e.g., photon bunching in thermal fields (as in intensity interferometry)

- Photons in the coherent state (e.g., a stable laser) are incident completely randomly

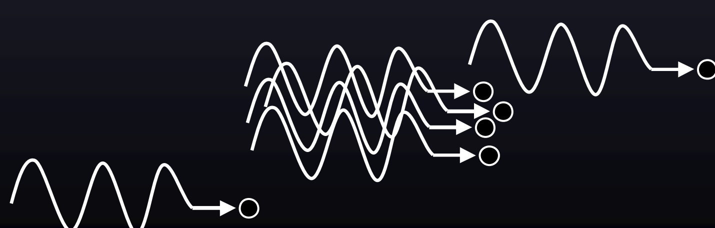


number-phase uncertainty principle can explain this

$\sigma_n^2 = \bar{n}$ “particle noise”

  - Two point-like photodetectors placed anywhere in the field record but random coincidence counts

- Photons in the thermal state (e.g., star light) are intrinsically bunched (true at any wavelength, optical or radio)



“wave noise”

$\sigma_n^2 = \bar{n} + \bar{n}^2$

(single-mode result)

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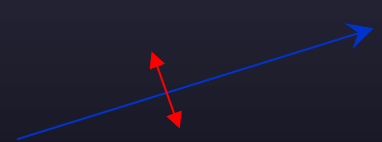
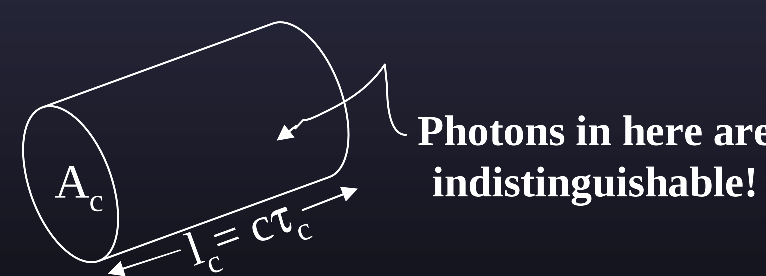
- Uncertainties related to detection
  - Do intrinsic QM uncertainties matter until we detect?
  - Imperfect quantum efficiency of detection + dark current + sky background + noise in the read-out process, as in a CCD/CMOS sensor

$$SNR \approx \eta\bar{N}/\sqrt{\eta\bar{N} + n_D + n_S + \sigma_R^2} \quad \text{(per pixel)}$$

- Photon counting vs. heterodyne mixing
  - **What is a mode?**
    - <u>Fourier interpretation</u> – field of single **k** and polarization **p**
    - <u>Spacetime interpretation</u> – field contained in a coherence volume (x-sectional area = coherence area, length = coherence length)





- Preferred viewpoint here is the latter one; corresponds to an elementary phase-space volume of order h^3,

$$(A_c\, c\tau_c)\,(\Delta k_x\, \Delta k_y\, \Delta k_z) \sim (2\pi)^3$$

- Photon counting statistics
  - Photons within a coherence volume obey single-mode statistics

    $\sigma_n^2 = \bar{n}$ (Coherent State)

    $\sigma_n^2 = \bar{n} + \bar{n}^2$ (Thermal state)

  - Detector area - A; Integration time - T; unit q. efficiency

(i) $A \ll A_c$, $T \ll \tau_c$ – the above expressions apply

(ii) $A \ll A_c$, $T \gg \tau_c$ – detection volume is sum of T/τ_c elementary coherence volumes; variances add

e.g., $T = 4\tau_c$

Coherent state: $\sigma_N^2 = \frac{\bar{N}}{M_t} \times M_t = \bar{N}, \quad M_t \equiv T/\tau_c$

Thermal state:

$$\sigma_N^2 = \left[\frac{\bar{N}}{M_t} + \left(\frac{\bar{N}}{M_t}\right)^2\right] \times M_t = \bar{N}(1 + \bar{N}/M_t)$$

Degeneracy parameter, δ_C

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- Degeneracy parameter, δ_c
  Max value = (A/A_c)·photons/mode, $\frac{A}{A_c} \cdot \frac{1}{\exp(h\nu/k_B T)-1}$

- Photon counting for thermal radiation - optical vs radio

$$\delta_c\,(\lambda = 0.5\mu m) \sim \frac{A}{A_c} \cdot 10^{-2} << 1$$ (Shot noise)

$$\delta_c\,(\lambda = 0.5\,m) \sim \frac{A}{A_c} \cdot 10^{6} > 1$$ (Wave noise dominated)

(E.g., for the VLA, $(A/A_c) \sim 10^{-4} – 10^{-5}$)

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- Photon counting the only practical approach at optical/near IR wavelengths ⇔ signal-dep. shot noise
  - Optical amplifiers cannot overcome shot noise – amplify both signal and noise plus add quantum noise ➔ SNR is actually reduced (*Prasad, 1994*)

- Heterodyning at far IR/radio – highly tunable, low-noise LO's available – direct amplitude detection
  - Voltage output of the ith antenna receiver (on correcting geometric delay)

$$\boxed{V_i(t) = \int [E_i(t') + N_i(t')]\; h_R(t-t')\, dt'}$$

Radiation field (Gaussian statistics)

Gaussian white noise

Receiver impulse response f'n (centered in freq at $\omega_{LO}+\omega_{IF}$)

  - Correlations,

$$\begin{aligned} \langle E_i(t) E_j^*(t')\rangle &= S\,\mu_{ij}\,\mu(t-t'), \\ \langle N_i(t) N_j^*(t')\rangle &= W_N \delta_{ij}\delta(t-t') \end{aligned}$$

  - Correlator output:

Low-pass filter

$$\boxed{C_{ij}(t) = \int V_i(t') V_j^*(t')\; h_C(t-t')\, dt'}$$

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- Noise make-up
  - **Optical:** Poisson shot noise, additive read-out noise (absent for highly cooled photodiodes), atmospheric turbulence, dome turbulence, etc
  - **Radio:** essentially additive Gaussian noise
    - Wave noise (or self noise), receiver noise, atmospheric emission noise, μ-wave background, ground radiation, etc – their FDs add
    - System temperature is a useful characterization of noise,

$$T_{sys} = T_{rx} + T_{sky} + T_{gr} + T_{bg} + T_{loss} + T_{cal}$$

    - Typically, $T_a \ll T_{sys}$, but sometimes $T_a \sim T_{sys}$ or even $T_a \gg T_{sys}$

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Basics of interferometric image synthesis

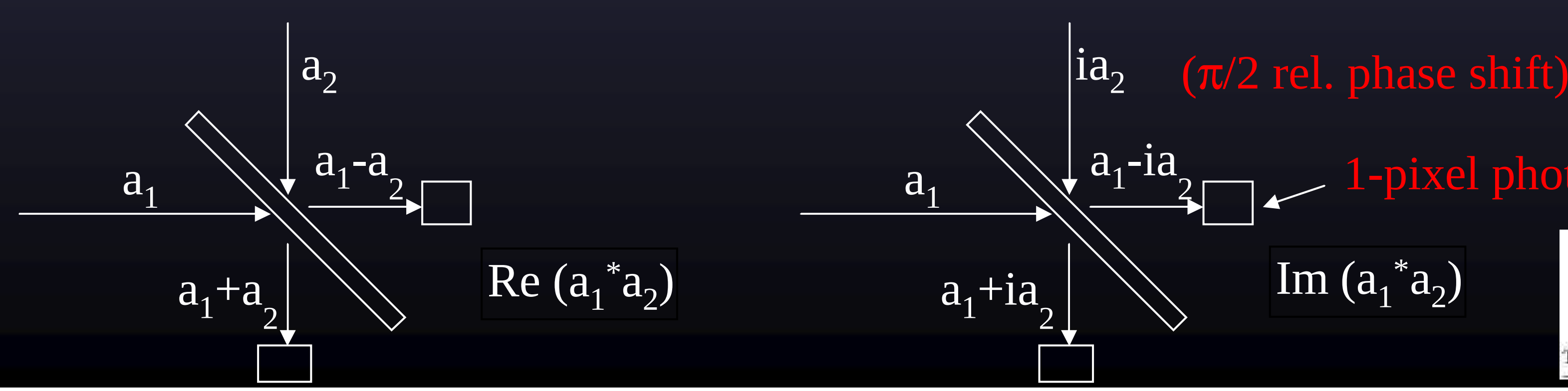
- Van Cittert-Zernike theorem:

  Image is the 2D FT of the corresponding spatial coherence function (valid for sufficiently small FOV), $I(\vec{\theta}) = \int d^2u \;\; \mu(\vec{u}) \,\mathrm{exp}(i2\pi\vec{u}\cdot\vec{\theta})$

- Discrete implementation – K complex visibility measurements

$$I_d(\vec{\theta}) = \mathrm{Re} \;\sum_k \mu(\vec{u}_k)\,\mathrm{exp}(i2\pi\vec{u}_k\cdot\vec{\theta})$$

- Measurement strategies
  - Photon counting at optical λ's – two common approaches
    - 4-point fringe sensing via difference measurements



- Fringes on an array detector, e.g., a high QE photodetector array – use of DFT as a fringe-visibility estimator,

$$\widehat{\mu}_{ij} = \sum_{\mathbf{p}} k_{ij}(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{u}_{ij}}$$

– Heterodyne mixing with LO at radio λ's

- Receiver voltage at the j^{th} antenna

$$V_j(t) = \int_{-\infty}^{\infty} [E_j(t') + N_j(t')]\, h_R(t-t')\, dt'$$

- Additive noise
- Impulse response f'n h_R is a bandpass filter centered at $(\nu_{LO}+\nu_{IF})$
- Correlator – can be regarded as a lowpass filter of product $V_j V^*_k$

$$C_{jk}(t) = \int_{-\infty}^{\infty} [V_j(t') V_k^*(t')]\, h_C(t-t')\, dt'$$

- A variety of nonlinear deconvolution methods used to create the final image from the estimates of μ_{jk}
- Noise/sensitivity to be analyzed for dirty image only

Noise analysis of optical interferometers

I. Intensity interferometer (HB-T)

II. Ideal Michelson interferometer

III. Ground based synthesis array

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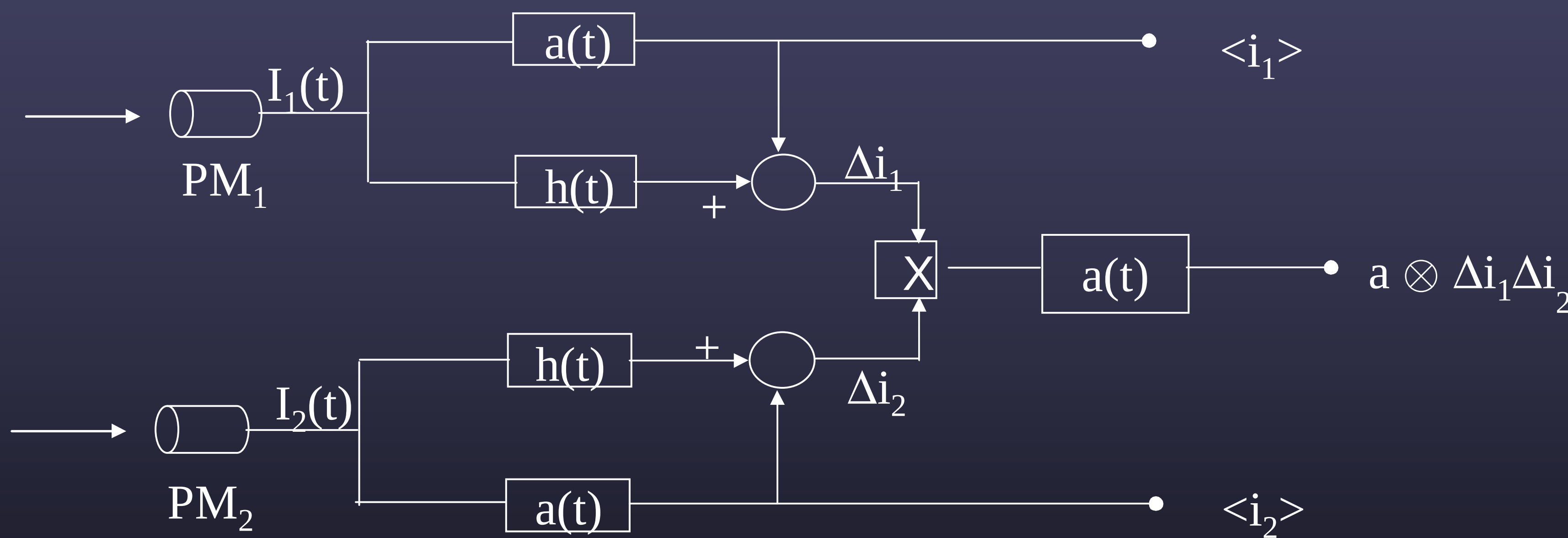
I. Intensity Interferometer

- Uses wideband photodetectors (PM's), wideband pre-filters [h(t)] and low-pass post-filters [a(t)] to improve SNR



- Exact Analysis:

$$I_1(t) = \frac{e\,n_1}{\Delta t_1}, \quad I_2(t) = \frac{e\,n_2}{\Delta t_2}$$

$$\begin{aligned} P(n_1, n_2 | E_1, E_2) &= \frac{(\alpha_1 |E_1|^2 \Delta t_1)^{n_1}}{n_1!} e^{-\alpha_1 |E_1|^2 \Delta t_1} \\ &\times \frac{(\alpha_2 |E_2|^2 \Delta t_2)^{n_2}}{n_2!} e^{-\alpha_2 |E_2|^2 \Delta t_2} \end{aligned}$$

$$P(n_1, n_2) = \langle P(n_1, n_2 | E_1, E_2) \rangle_{E_1, E_2}$$

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- Mean signal: $\langle C_I \rangle = \langle \Delta i_1 \Delta i_2 \rangle \sim \frac{B_{pre}}{B_{post}} \, |\mu_{12}|^2$ **No photon bias!!**
- Fluctuation:

$$\Delta C_I(t) \equiv \int_{-\infty}^{\infty} a(t-\xi)[\Delta i_1(\xi)\Delta i_2(\xi) - \langle \Delta i_1 \Delta i_2 \rangle] d\xi$$

- Variance: $\sigma_I^2 = \langle [\Delta C_I(t)]^2 \rangle$
- SNR: $\frac{\langle C_I \rangle}{\sigma_I} \approx \frac{|\mu_{12}|^2}{\sqrt{1+|\mu_{12}|^4}} \sqrt{\frac{B_{pre}}{B_{post}}}$ **Classically Correct!!**

- Standard features contained in this <u>classical expression</u>

  # coherent samples: $\frac{B_{pre}}{B_{post}}$

  * SNR per sample < 1 (intensity fluctuations for thermal fields)

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II. Ideal Michelson Interferometer (Prasad, Kulkarni, 1989)

- For example, space/lunar based synthesis arrays
- Path length differences can in principle be maintained with high accuracy

  ➔ Field-amplitude correlations can be directly inferred.
- For **pairwise beam combination** on a P-pixel array detector, the DFT estimator is

dc component

$$\widehat{\mu}_{ij} = \sum_{\mathbf{p}} k_{ij}(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{u}_{ij}}, \qquad \widehat{\mu}_{ij}^{(0)} = \sum_{\mathbf{p}} k_{ij}(\mathbf{p})$$

where $\langle k_{ij}(\mathbf{p})\rangle = 2K_0[1+\gamma_{ij}\cos(\mathbf{p}\cdot\mathbf{u}_{ij}+\phi_{ij})]$: mean count

*Assume no detector noise ➔ only photon shot noise

$\langle \Delta k_b(\mathbf{p})\ \Delta k_{b'}(\mathbf{p}')\rangle = \delta_{\mathbf{bb}'}\delta_{\mathbf{pp}'}\langle \mathbf{k_b}(\mathbf{p})\rangle$, b, b′: baselines

*Mean # photons detected by array per frame,

$$\langle C\rangle = \underbrace{2K_0}_{2\langle N\rangle} P\cdot n(n-1)/2$$

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* Estimator of the dirty map:

Negative freq's

$$\hat{I}(\mathbf{q}) = \frac{1}{2}\sum_{\mathbf{b}}\left\{[\hat{\mu}_{\mathbf{b}}\ e^{i\mathbf{q}\cdot\mathbf{u}_{\mathbf{b}}} + \text{ c.c.}] + f\,\mu_{\mathbf{b}}^{(0)}\right\}$$

dc components

(a) If the dc components are dropped (f=0), then

Mean image: $\langle \hat{I}_0(\mathbf{q})\rangle = \langle N\rangle \mathrm{Re}\left(\sum_{\mathbf{b}} \mu_{\mathbf{b}}\ e^{i\mathbf{q}\cdot\mathbf{u}_{\mathbf{b}}}\right)$

Variance:

$$\begin{aligned}
\mathrm{var}[\hat{I}_0(\mathbf{q})] &= \frac{1}{4}\sum_{\mathbf{b}}\sum_{\mathbf{b}'}[\underbrace{\langle\Delta\hat{\mu}_{\mathbf{b}}\Delta\hat{\mu}_{\mathbf{b}'}\rangle}_{0} e^{i\mathbf{q}\cdot(\mathbf{u}_b+\mathbf{u}_{b'})} \\
&+ \underbrace{\langle\Delta\hat{\mu}^*_{\mathbf{b}}\Delta\hat{\mu}_{\mathbf{b}'}\rangle}_{2\langle N\rangle\delta_{\mathbf{bb}'}} e^{i\mathbf{q}\cdot(\mathbf{u}_b-\mathbf{u}_{b'})} + \text{ c.c.}] \\
&= \langle N\rangle n(n-1)/2 = \langle C\rangle/2
\end{aligned}$$

Uniform across the image!

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(b) If the dc components are fully included (f=1), then

Mean image: $\langle \hat{I}_1(\mathbf{q}) \rangle = \langle N \rangle \mathrm{Re}\left[\sum_{\mathbf{b}} \left(1 + \mu_{\mathbf{b}}\ e^{i\mathbf{q}\cdot\mathbf{u}_{\mathbf{b}}} \right) \right] \geq 0$

Variance:
$$\begin{aligned}
\mathrm{var}[\hat{I}_1(\mathbf{q})] &= \overbrace{\mathrm{var}[\hat{I}_0(\mathbf{q})]}^{\langle C \rangle/2} \\
&+ \frac{1}{4} \sum_{\mathbf{b},\mathbf{b}'} [\underbrace{\langle \Delta\hat{\mu}_{\mathbf{b}} \Delta\hat{\mu}^{(0)}_{\mathbf{b}'} \rangle}_{\langle N \rangle \mu_{\mathbf{b}} \delta_{\mathbf{b}\mathbf{b}'}} e^{i\mathbf{q}\cdot\mathbf{u}_b} + \text{ c.c} \\
&+ \underbrace{\langle \Delta\hat{\mu}^{(0)}_{\mathbf{b}} \Delta\hat{\mu}^{(0)}_{\mathbf{b}'} \rangle}_{2\langle N \rangle \delta_{\mathbf{b}\mathbf{b}'}}] \\
&= \langle C \rangle/4 + \langle \hat{I}_1(\mathbf{q}) \rangle
\end{aligned}$$

Non-uniform, image-dependent across the image!

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(c) **Dirty-Image SNR:** consider a point source at phase center

(i) Without dc – $$\frac{\langle \hat{I}_0(0)\rangle}{\sqrt{\mathrm{var}[I_0(0)]}} = \left(\frac{\langle C\rangle}{2}\right)^{1/2}$$

(ii) With dc – $$\frac{\langle \hat{I}_1(0)\rangle}{\sqrt{\mathrm{var}[I_1(0)]}} = \left(\frac{8}{5}\right)^{1/2}\left(\frac{\langle C\rangle}{2}\right)^{1/2}$$

(d) **Summary Remarks:**

1. Without dc, variance is uniform across image; with dc, variance is image dependent (due to Poisson-correlated dc–non-dc components on each photodetector)
2. With dc, the SNR is enhanced in general
3. Similar results even when fringes are not detected pairwise

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Remarks:

(1) SNR of an amplitude interferometer under ideal conditions of Poisson statistics is essentially of order $\sqrt{\frac{\langle C \rangle}{2}}$.

(2) The exact nature of the unbiased visibility estimator is not important, nor is the fringe-detection geometry nor the number of pixels in the detectors nor the degree of baseline redundancy.

(3) Inclusion of zero-freq component is needed for proper image reconstruction; also tends to improve image SNR.

(4) For *m* independent frames of visibility data, the SNR is expected to exhibit the usual $m^{1/2}$ fold improvement → SNR only depends on the total # photons detected by the array.

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(5) These conclusions are modified when detector noise is present, as in CCD arrays. Single-pixel detectors are clearly preferable in that case.

(6) Deconvolution generates interpolated spatial freq's based on both measurements and prior knowledge/constraints → photons are simply redistributed and SNR should be essentially the same in the final synthesis image

(7) The same form for the SNR obtains for *any* observable in the high-flux limit provided detection is limited only by photon noise.

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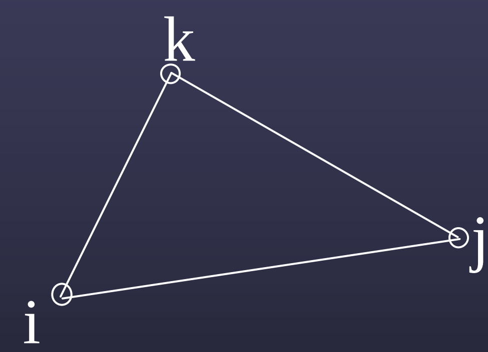
Bispectrum-Based Optical Image Synthesis

- For very faint sources, self-cal won't work; triple correlations may be the only recourse.

$\hat{B}_{ijk} = \hat{\mu}_{ij} \cdot \hat{\mu}_{jk} \cdot \hat{\mu}_{ki}$, independent of antenna phases

Phase – closure phase,



$$\phi_{ijk} = \phi_{ij} + \phi_{jk} + \phi_{ki}$$

Amplitude – estimated via power spectra,

$$|B_{ijk}| = \sqrt{|\mu_{ij}|^2 |\mu_{jk}|^2 |\mu_{ki}|^2}$$

* Only (n-1)(n-2)/2 indep. closure phases for n-element array, but n(n-1)(n-2)/6 indep. measurements → hybrid mapping approaches must be used for deconvolution

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* Assume pair-wise beam combination for simplicity

- Power spectrum, p_{ij}:

$$\widehat{\mu}_{ij} = \sum_{\mathbf{p}} k_{ij}(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{u}_{ij}}, \qquad \widehat{p}_{ij} = |\widehat{\mu}_{ij}|^2$$

Mean: $\langle \widehat{p}_{ij} \rangle = |\mu_{ij}|^2 \langle N \rangle^2 + 2\langle N \rangle$ Photon bias

Unbiased estimator: $\widehat{q}_{ij} = \widehat{p}_{ij} - \sum_p k_{ij}(p)$

Mean – $\langle \widehat{q}_{ij} \rangle = |\mu_{ij}|^2 \langle N \rangle^2$

Variance – very complicated expression, but simplifies for pair-wise fringe detection

SNR – $$\frac{|\mu_{ij}|^2 \langle N \rangle}{2\sqrt{1 + |\mu_{ij}|^2 \langle N \rangle}} \rightarrow \sqrt{\frac{|\mu_{ij}|^2 \langle N \rangle}{2}} \text{ for } |\mu_{ij}|^2 \langle N \rangle >> 1$$

$$\rightarrow \frac{|\mu_{ij}|^2 \langle N \rangle}{2} \text{ for } |\mu_{ij}|^2 \langle N \rangle << 1$$

Familiar high-flux result

- **Bispectrum, B_{ijk}:** $\widehat{B}_{ijk} = \widehat{\mu}_{ij}\widehat{\mu}_{jk}\widehat{\mu}_{ki}$

Mean: $\langle \widehat{B}_{ijk} \rangle = \langle N \rangle^3 \mu_{ij} \cdot \mu_{jk} \cdot \mu_{ki} = \langle N \rangle^3 \gamma_1 \gamma_2 \gamma_3 \exp(i\psi_{ijk})$

Unbiased due to pair-wise beam combination

SNR:

$$\frac{\gamma_1\gamma_2\gamma_3\langle N\rangle^{3/2}}{[2\langle N\rangle^2(\gamma_1^2\gamma_2^2+\gamma_2^2\gamma_3^2+\gamma_3^2\gamma_1^2)+4\langle N\rangle(\gamma_1^2+\gamma_2^2+\gamma_3^2)+8]^{1/2}}$$

(take all γ's equal)

$$\rightarrow \sqrt{\frac{\gamma^2\langle N\rangle}{6}} \text{ for } \gamma^2\langle N\rangle >> 1$$

$$\rightarrow \frac{(\gamma^2\langle N\rangle)^{3/2}}{2\sqrt{2}} \text{ for } \gamma^2\langle N\rangle << 1$$

Familiar high-flux result

- In low-flux limit, use all bispectra to synthesize image

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- Synthesized image SNR – no closed-form expression (reconstruction is nonlinear and iterative)
- Toy Example: Point-source flux, $\hat{S} = (\hat{F})^{1/3}, \quad \hat{F} = \mathrm{Re} \sum_{\langle ijk \rangle} \hat{B}_{ijk}$

  Mean:

$$\langle \hat{F} \rangle = \gamma^3 \langle N \rangle^3 n_t, \quad n_t \equiv n(n-1)(n-2)/6$$

  Variance – composed of variances of the bispectra and covariances of bispectra with one common side

  SNR (S) $\approx$ 3· SNR(F)

(γ: instrumental decorrelation)

$$\frac{\gamma(\langle C \rangle/2)^{1/2}}{\sqrt{1 + (6\gamma^2 \langle N \rangle + 4)/[3(n-2)\gamma^4 \langle N \rangle^2]}}$$

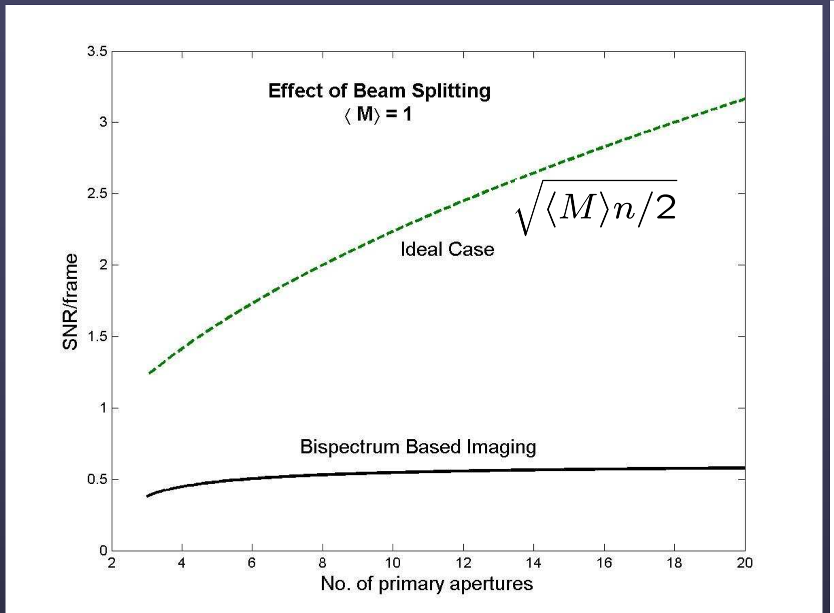
$$\rightarrow \sqrt{\gamma^2 \langle C \rangle / 2} \qquad \text{for } \gamma^2 \langle N \rangle n >> 1$$

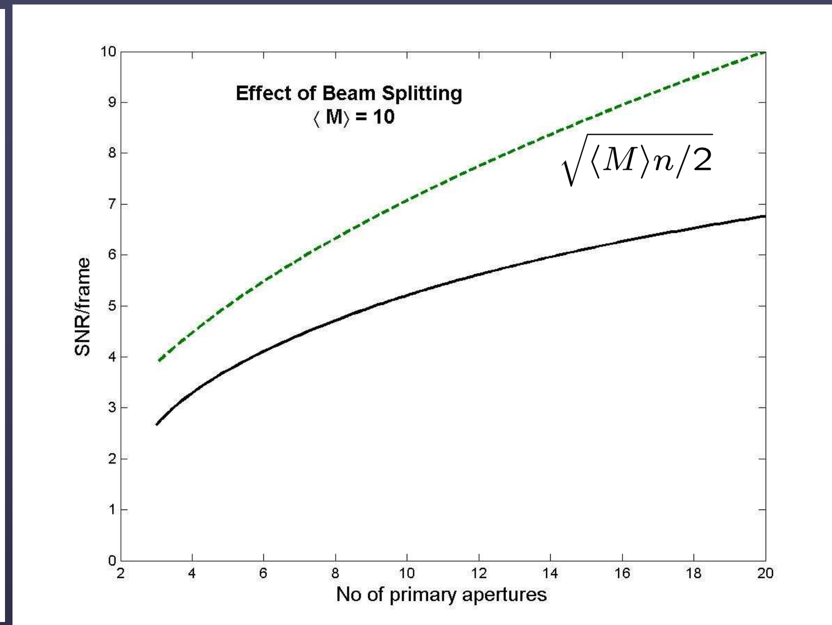
$$\rightarrow \sqrt{3/8}\ \gamma^3 (n \langle N \rangle)^{3/2} \quad \text{for } \gamma^2 \langle N \rangle n << 1$$

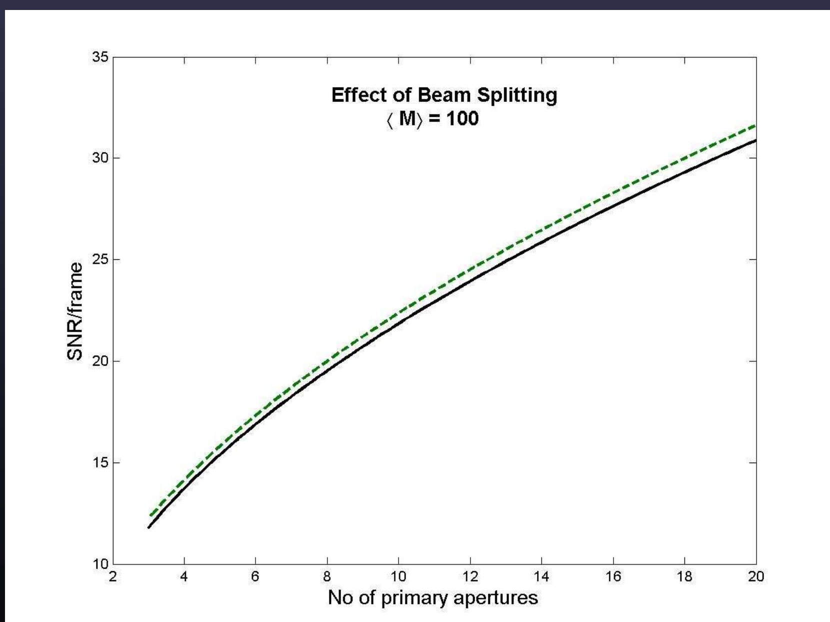
Matches high-flux result for ideal observation

Depends only on source strength

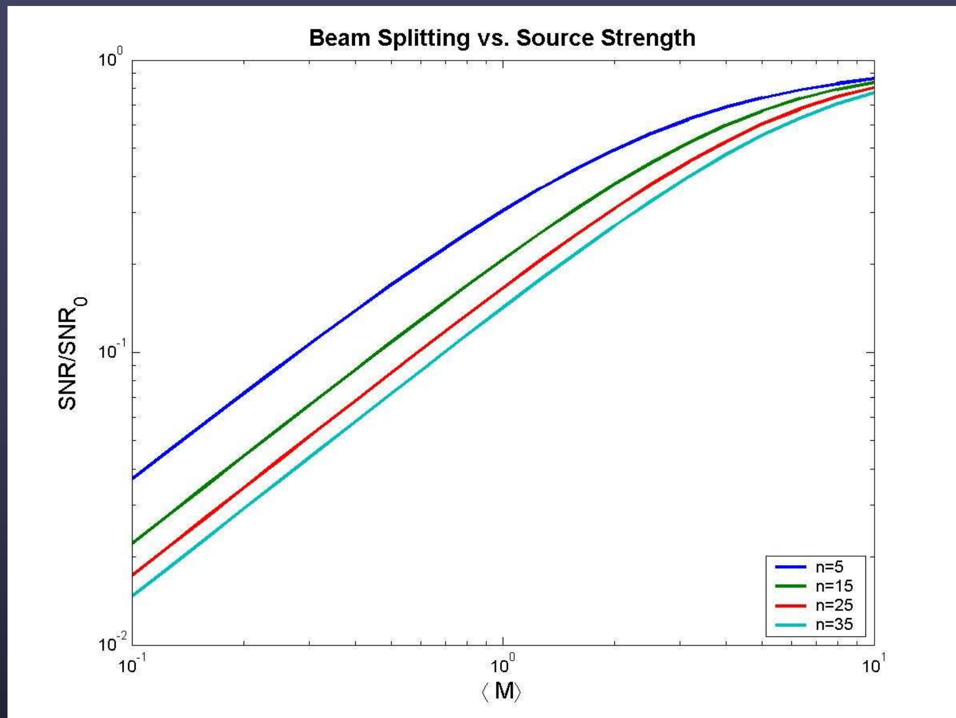
Bispectrum vs Ideal Image Synthesis







- <M> = <C> / n ~ source strength
- For weak sources, beam splitting for pair-wise combination is catastrophic
- Situation greatly exacerbated when read noise is present



SNR penalty quite severe for large array size, n >> 1, and weak sources, <M> <<1

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Noise analysis of radio interferometers

I. Ideal Michelson interferometer

II. Ground based synthesis array

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- Voltage output of the i^{th} antenna receiver (after correcting geometric delay) – correlator input with Gaussian statistics,

$$V_i(t) = \int \left[E_i(t') + N_i(t')\right]\, h_R(t-t')\, dt'$$

- Mean voltage correlations:
  - Instantaneous input:

Noise PSD

$$\langle V_i(t) V_j^*(t)\rangle = \left(W_S\, \mu_{ij} + W_N \delta_{ij}\right)\, B_R,\; W_S \equiv S/B_S$$

  - Integrated output:

Signal PSD

$$\langle \widehat{C}_{ij}\rangle = \langle V_i(t) V_j^*(t)\rangle \otimes h_C(t) = \left(W_S\, \mu_{ij} + W_N \delta_{ij}\right)\, \frac{B_R}{B_C}$$

No. of coherent samples (ratio of pre and post bandwidths)

- Fluctuations of voltage correlations,

$$\sigma_{ij}^2 \equiv \langle |\Delta \widehat{C}_{ij}|^2\rangle = (W_S + W_N)^2\, \frac{B_R}{B_C}$$

- (Pseudo) SNR,

$$\chi_{ij} \equiv \frac{|\langle \widehat{C}_{ij}\rangle|}{\sigma_{ij}} = \frac{W_S\, |\mu_{ij}|}{(W_S + W_N)} \sqrt{\frac{B_R}{B_C}}$$

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Covariances of Two Different Voltage Correlations

- Use complex correlations – composed of correlations of real and imaginary parts
- Two baselines with a common antenna

$$\begin{aligned} \text{cov}\left(C_{ij}, C_{ik}\right) &\equiv \langle C_{ij} C_{ik} \rangle - \langle C_{ij} \rangle \langle C_{ik} \rangle \\ &= W_S^2 \cdot \mu_{ij} \cdot \mu_{ik} \cdot \frac{B_R}{B_C} \end{aligned}$$

$$\text{cov}\left(C_{ij}, C_{ki}\right) = W_S (W_S + W_N) \cdot \mu_{kj} \cdot \frac{B_R}{B_C}$$

- Two disjoint baselines

$$\text{cov}\left(C_{ij}, C_{k\ell}\right) = W_S^2 \cdot \mu_{i\ell} \cdot \mu_{kj} \cdot \frac{B_R}{B_C}$$

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Noise Analysis for Ideal Synthesis Imaging

- Dirty image estimate:

$$\hat{I}(\mathbf{q}) = \sum_{i=1}^{n}\sum_{j\neq i}^{n} \hat{C}_{ij} e^{i\mathbf{q}\cdot\mathbf{u}_{ij}} + \frac{1}{n}\sum_{i=1}^{n} \hat{C}_{ii}$$

dc component added with natural weight

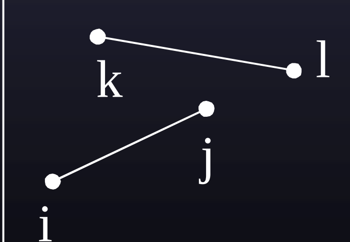
- Mean:

$$\langle \hat{I}(\mathbf{q}) \rangle = \frac{B_R}{B_C}\left[W_S \left(1 + \sum_{i=1}^{n}\sum_{j\neq i}^{n} \mu_{ij} e^{i\mathbf{q}\cdot\mathbf{u}_{ij}} \right) + W_N \right]$$

dc component

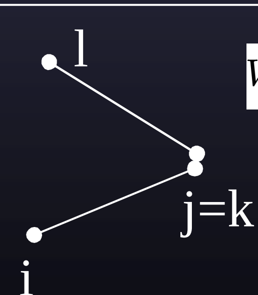
Bias, easily subtracted

- Variance:
  - Consists of a large number of covariances, two shown below



$$W_S^2 \cdot \mu_{i\ell} \cdot \mu_{kj} \cdot \frac{B_R}{B_C}$$

i≠ j≠ k≠ l   n(n-1)(n-2)(n-3) terms



$$W_S(W_S + W_N) \cdot \mu_{i\ell} \cdot \frac{B_R}{B_C}$$

i≠ j≠ l   n(n-1)(n-2) terms

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- Toy example – a point source at the phase center, $\mu_{ij} = 1$ *for all i, j*
  - SNR at the map center, **q** = 0

Mean: $\langle \hat{I}(\mathbf{0}) \rangle = \frac{B_R}{B_C} \cdot W_S \cdot (n^2 - n + 1)$

Variance, SNR – general expressions quite involved

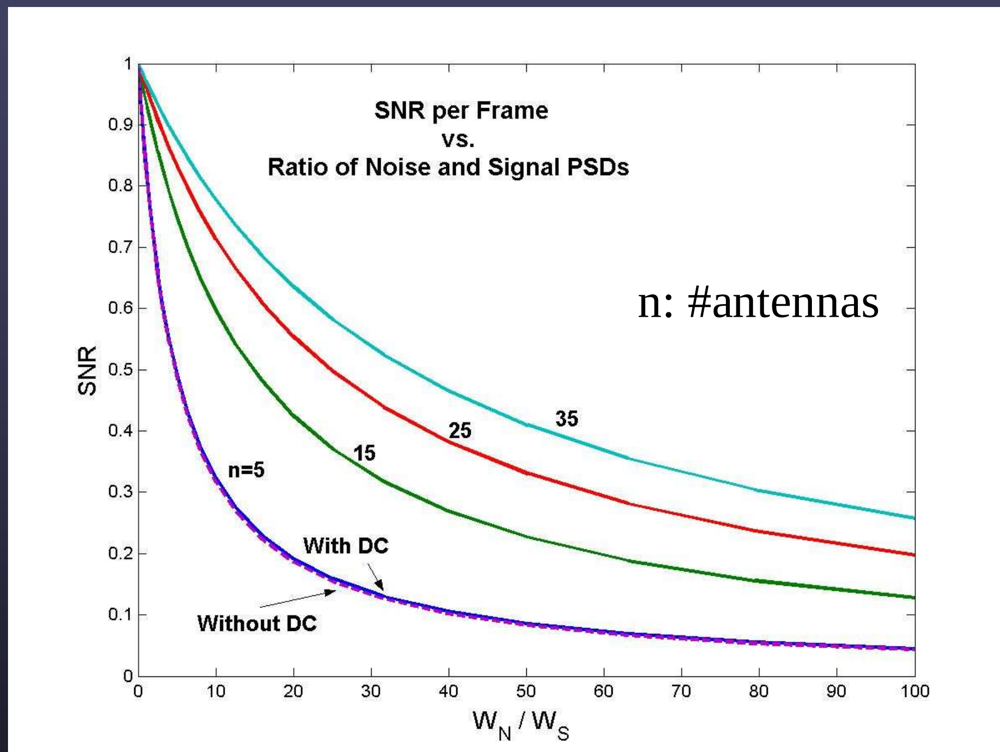
Strong-source limit, $W_S >> W_N$: $SNR = \sqrt{\frac{B_R}{B_C}}$, indep. of n

Weak-source limit, $W_S << W_N$:

$$SNR = \sqrt{\frac{B_R}{B_C}} \cdot \frac{W_S}{W_N} \cdot \frac{n^2 - n + 1}{\sqrt{n^2 - n + 1/n}}$$

The Point-Source Case

Arbitrary W_N / W_S:



- Hardly any change when DC component is excluded
- Graceful decline of SNR with increasing noise, for large n
- Even for $1 \leq W_N/W_S \leq n$, SNR is still above 0.5 – unlike signal, noise adds incoherently across the array

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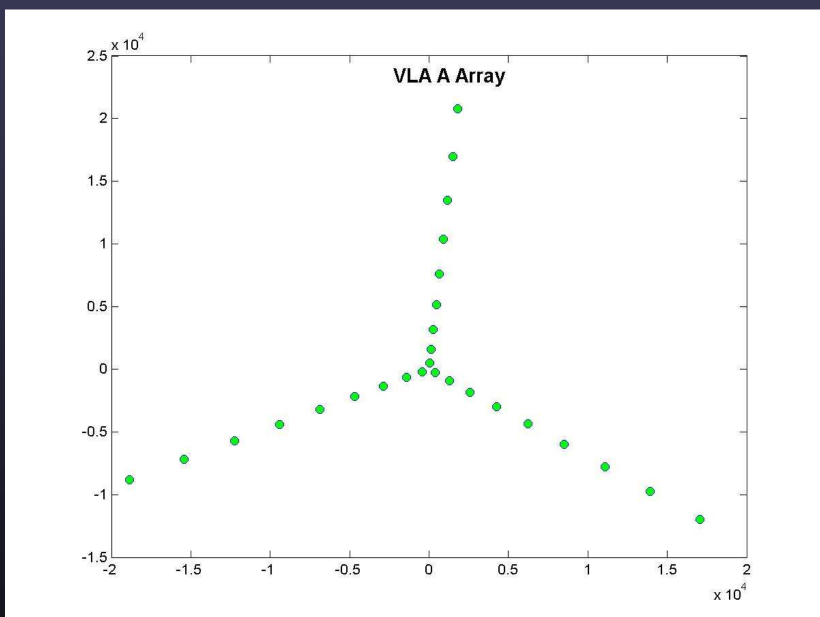
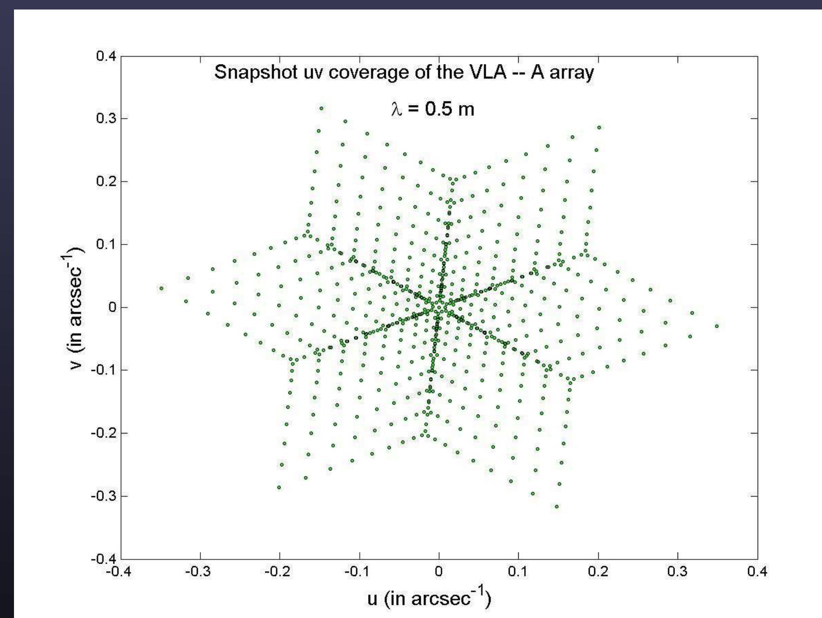
- Real Example: Cygnus A Imaged by a VLA Type Ideal Interferometer

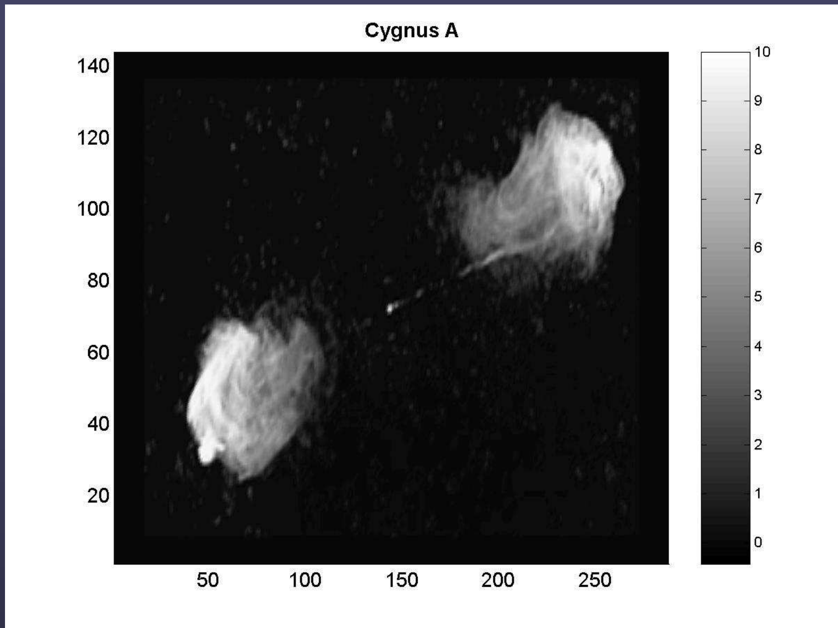


VLA A Array (all distances in m)

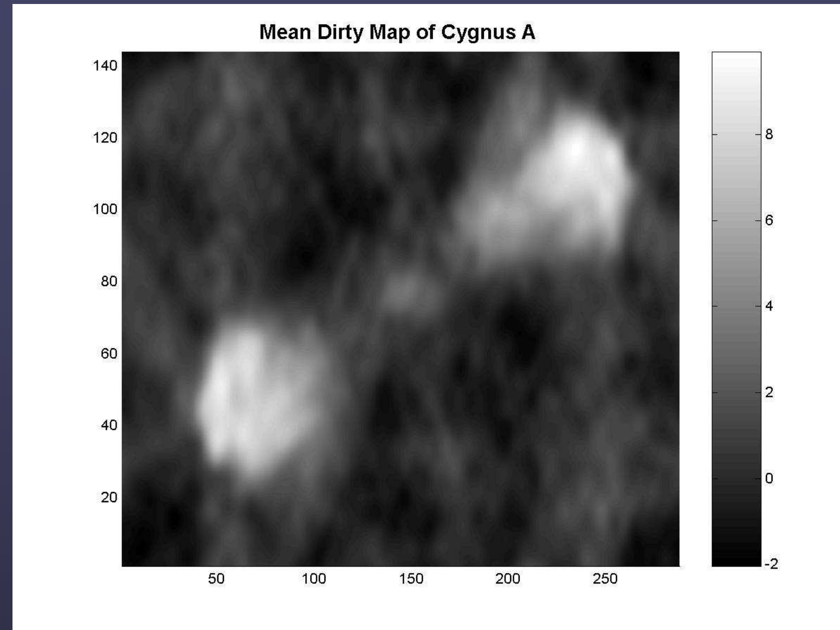


Nearly nonredundant, snapshot uv coverage of the A Configuration

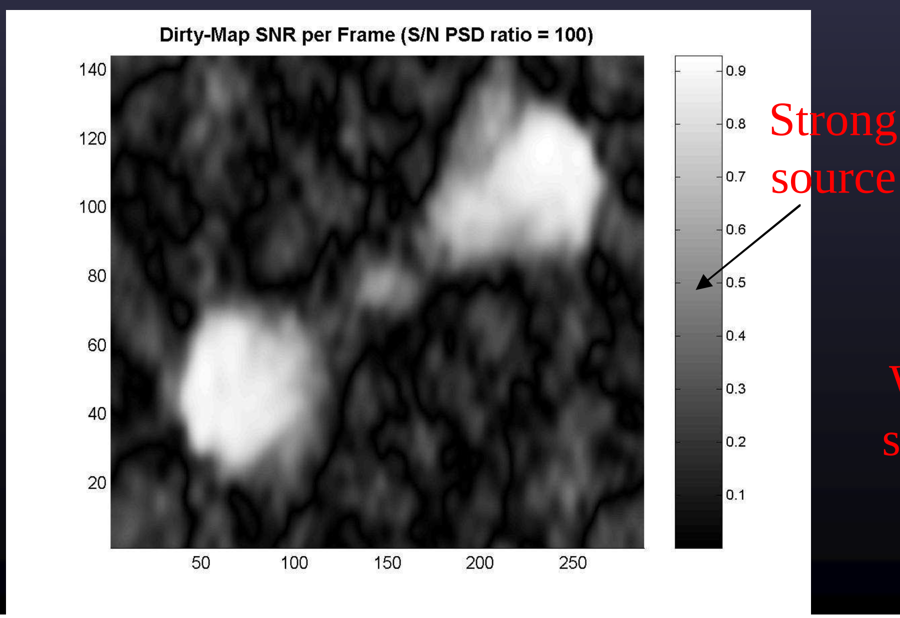
High-Res. Image ("Truth")

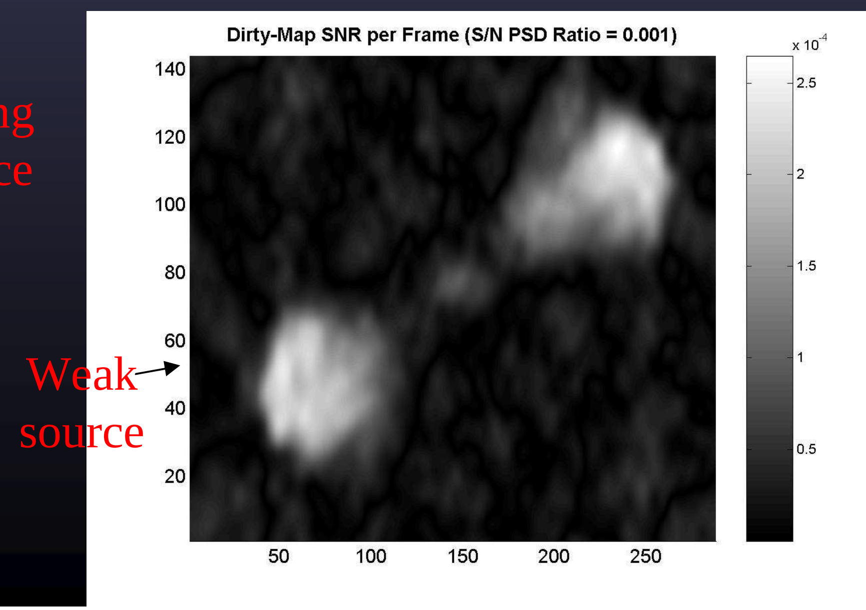


Dirty Map

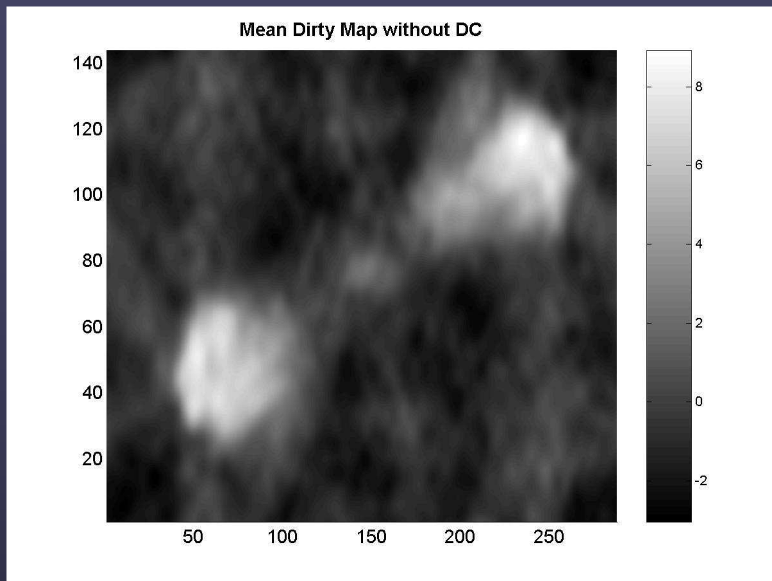


SNR in Dirty Map when DC Component is Included

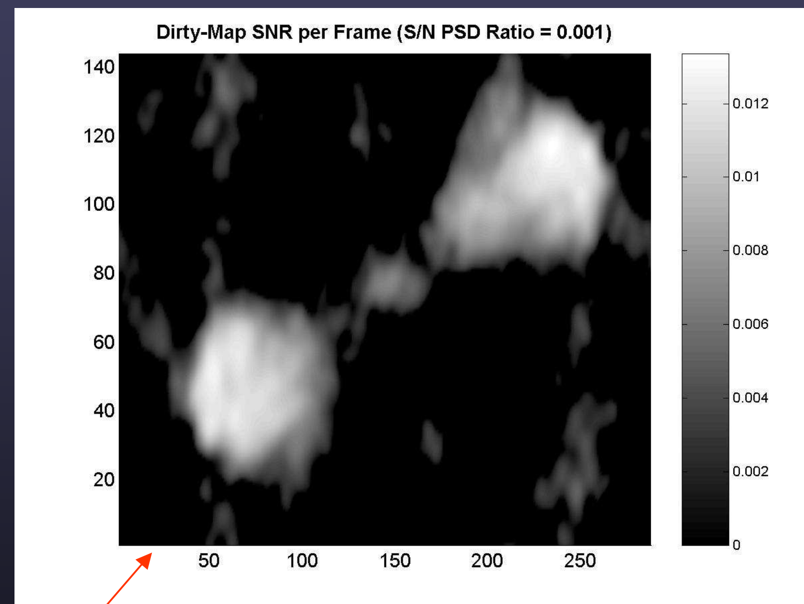




Dirty Map

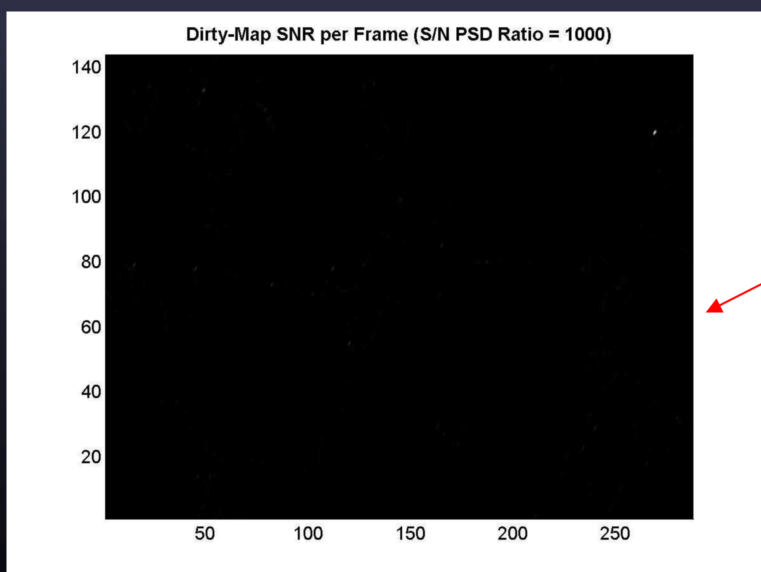


SNR in Dirty Map when DC Component is Excluded



Weak source

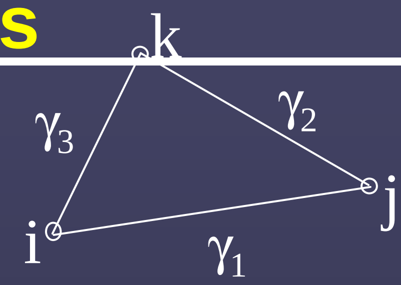
Uniform Noise Variance



Strong source

Signal-Dependent Noise

Noise Analysis of Bispectrum-Based Image Synthesis



- **Bispectrum estimate:**

$$\widehat{B}_{ijk} = \widehat{\mu}_{ij} \cdot \widehat{\mu}_{jk} \cdot \widehat{\mu}_{ki}$$

  – Mean:

ignored

$$\langle \widehat{B}_{ijk} \rangle = (B_R/B_C)^3\, W_S^3 \cdot \mu_{ij} \cdot \mu_{jk} \cdot \mu_{ki} + O(B_R/B_C)^2 \text{ terms}$$

  – Pseudo-variance:

$$\langle |\Delta \widehat{B}_{ijk}|^2 \rangle \;=\; \left(\frac{B_R}{B_C}\right)^5 W_S^6 \Big[(1+\chi)^2 (|\gamma_1^2\gamma_2^2 + \gamma_2^2\gamma_3^2 + \gamma_3^2\gamma_1^2) + 6\gamma_1^2\gamma_2^2\gamma_3^2 \Big], \quad \chi \equiv W_N/W_S$$

  – SNR:

$$\sqrt{\frac{B_R}{B_C}} \cdot \frac{\gamma_1\gamma_2\gamma_3}{[(1+\chi)^2(\gamma_1^2\gamma_2^2+\gamma_2^2\gamma_3^2+\gamma_3^2\gamma_1^2)+6\gamma_1^2\gamma_2^2\gamma_3^2]^{1/2}}$$

$$\to \sqrt{\frac{B_R}{B_C}} \frac{\gamma}{\sqrt{3(1+2\gamma^2)}} \text{ for } W_S >> W_N$$

$$\to \frac{\gamma W_S}{3 W_N} \sqrt{\frac{B_R}{B_C}} \text{ for } W_S << W_N$$

(all γ's equal)

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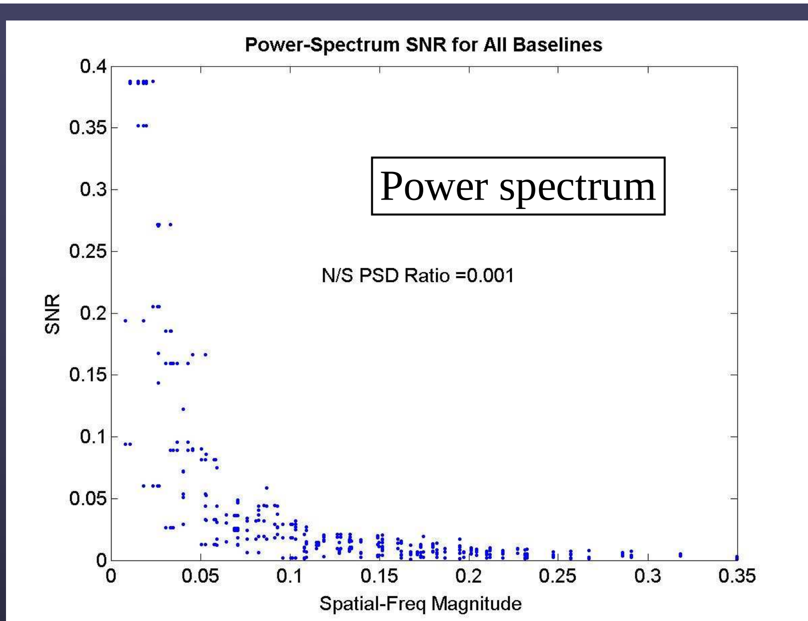
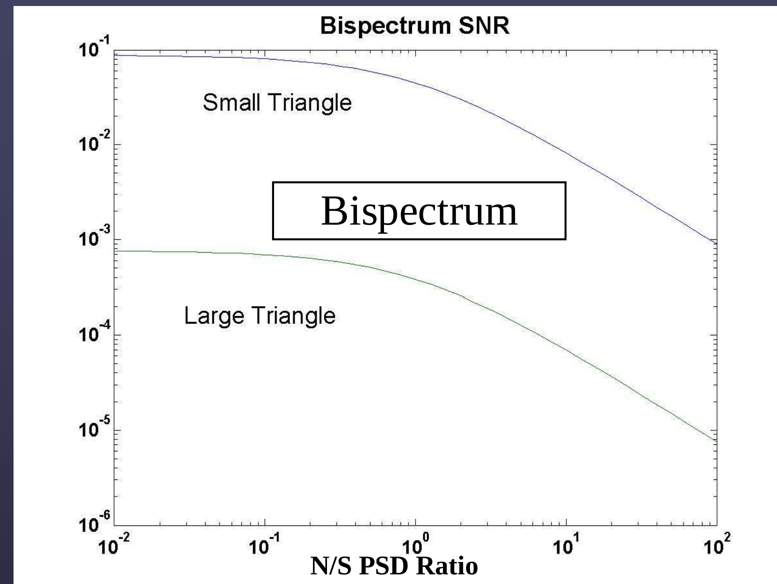
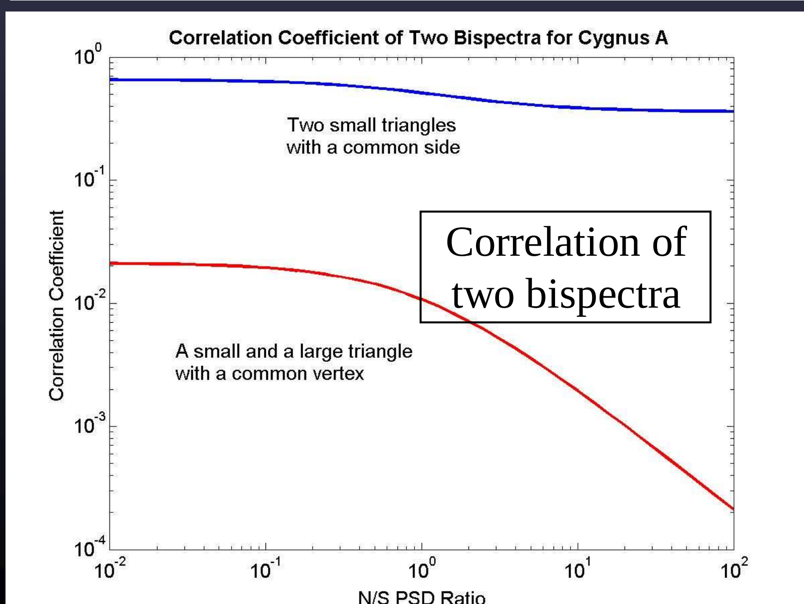
Various SNR's for Cygnus A Observed on VLA-A







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- Synthesized image SNR – no closed-form expression (reconstruction is nonlinear and iterative)
- Toy Example: Point-source flux, $\hat{S} = (\hat{F})^{1/3}, \quad \hat{F} = \text{Re} \sum_{\langle ijk \rangle} \hat{B}_{ijk}$

Mean: $\langle \hat{F} \rangle = \left(\frac{B_R}{B_C} W_S \right)^3 \cdot n_t, \quad n_t \equiv n(n-1)(n-2)/6$

Variance – composed of bispectrum variances and covariances with one and two common vertices

SNR (S) $\approx 3 \cdot$ SNR(F)

$$\sqrt{\frac{B_R}{B_C}} \frac{\sqrt{n(n-1)(n-2)/2}}{\sqrt{(n-2)(1+\chi)^2 + (n-2)(n-3)(1+\chi) + (n-3)(n^2-2n+4)/2+6}}$$

$$\to \sqrt{\frac{B_R}{B_C}} \qquad \text{for } \chi \equiv W_N/W_S << 1$$

$$\to \sqrt{\frac{B_R}{B_C}} \sqrt{n_b} \frac{W_S}{W_N} \qquad \text{for } \chi \equiv W_N/W_S >> 1$$

Matches high-flux result for ideal observation

Proportional to source strength

Concluding Remarks

- Overall noise has quantum mechanical, EM wave, and detection related constituents

- Optical synthesis imaging – without detector noise, SNR/frame scales as $(n\langle M\rangle)^{1/2}$ for direct amplitude correlation, as $\langle M\rangle^{3/2}$ for bispectral imaging, and as 1 for intensity interferometry (in the high-flux limit)

- Radio synthesis – additive receiver noise and Gaussian field statistics ➔ SNR/frame is indep of signal at high flux, $W_S \gg W_N$, and $\sim W_S/W_N$ at low fluxes, $W_S \ll W_N$

- SNR $\sim \sqrt{m}$, where m : # coherent samples

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