Astronomy 12 — Spring 1999 (S.T. Myers)

Homework Set 9

Due Thursday April 22, 1999 (5pm)

1. There are different distances in an expanding universe. The *comoving distance* is the distance measured in units that expand along with the Universe. By following the path of a photon in the Robertson-Walker metric for flat space ($k = 0$), we can obtain the relation for the comoving distance

$$\chi = \int_{t_1}^{t_0} \frac{c\,dt}{R}.$$

Use the Friedmann equation

$$\dot{R}^2 = \frac{8\pi G\rho}{3} R^2$$

to find dt, with

$$R(z) = \frac{R_0}{1+z} \qquad \rho_0 = \rho\, R^3$$

where $R_0 \equiv 1$, to obtain the formula

$$\chi(z) = \frac{2c}{H_0}\left[1 - (1+z)^{-1/2}\right].$$

For $H_0 = 100h$ km/s/Mpc, what is the comoving distance to the "big bang" at $z = \infty$?

2. A medium-sized cluster of galaxies like the Virgo cluster that we used in the previous problem set has a diameter of $D = 1.8\,h^{-1}$ Mpc. If this cluster were to keep the same *comoving size*, ie. it were to expand with the Universe

$$D_{com} = D = 1.8\,h^{-1} \text{ Mpc}$$

then its angular diameter would be given by the relation

$$\theta_\chi = \frac{D}{\chi}.$$

Show that this *comoving angular size* reaches a finite limiting angular diameter at $z = \infty$, and calculate this angular diameter for our $1.8\,h^{-1}$ comoving Mpc cluster, in arcseconds.

3. Consider now a galaxy of a fixed physical angular diameter of $D = 0.1\,h^{-1}$ Mpc. This will not in general expand with the universe, and thus in comoving coordinates it will appear to shrink with the scale factor

$$D_{com} = (1+z)\,D$$

so its apparent size will follow

$$\theta = \frac{(1+z)\,D}{\chi} = \frac{D}{d_A}$$

where we have used this to define the *angular diameter distance* d_A

$$d_A = \frac{\chi}{(1+z)}$$

which is true for all cosmologies.

Show that for our "flat" universe χ from the previous problem

$$d_A = \frac{2c}{H_0}\left[(1+z)^{-1} - (1+z)^{-3/2}\right]$$

and thus the apparent angular diameter θ reaches a minimum value θ_{min} at a finite redshift z_{min}. Compute z_{min} and θ_{min} in arcseconds for the galaxy. (Hint: find where $d\theta/d(1+z) = 0$.) Note that this means that in a flat universe, very distant objects need not appear very small!

4. What is the current *critical density* for the universe $\rho_{cr,0}$ in units of $M_\odot/\mathrm{Mpc}^3$ for $H_0 = 100\,h$ km/s/Mpc?

   Calculate the current critical energy density ($E = mc^2$)

$$u_{cr} = \rho_{cr} c^2 = \frac{3H_0^2 c^2}{8\pi G}$$

   for $H_0 = 100\,h$ km/s/Mpc.

   Calculate the energy density in the cosmic background radiation

$$u_{rad} = aT^4 = \frac{4\sigma}{c} T^4$$

   for $T_0 = 2.73$ K.

   What fraction of the critical density $\Omega_{rad} = u_{rad}/u_{cr}$ does the cosmic microwave background contribute? (Note: these will be parameterized by h.)

   If density evolves as $\rho = \rho_0(1+z)^3$ and temperature evolves as $T = T_0(1+z)$, at what redshift z_{cr} will the energy density of the radiation equal the critical density ($\Omega_{rad} = 1$)? Note that this is approximately the redshift where matter and radiation have equal "densities". (Hint: this is an estimate, so don't worry about keeping the total energy density equal to the critical density.)

5. The Coma cluster of galaxies is even more impressive than the Virgo cluster, and has a radial velocity dispersion of $\sigma_r = 977 km/s$. What is the true 3-D velocity dispersion of the galaxies σ_v?

   If the virial radius of the Coma cluster is $1.8\,h^{-1}$ Mpc, what is the implied total virial mass of the cluster (in $M_\odot$)? Be sure to express this in terms of the Hubble parameter h.

   If the absolute visual magntitude of the Coma cluster is $M_V = -26.92$, what is the visual luminosity of Coma (in $L_\odot$)? What is the implied mass-to-light ratio of the Coma cluster?

   If an average galaxy in the Coma cluster has a mass-to-light ratio typical of spiral or normal elliptical galaxies ($\sim$ 10 msun/lsun), what is the total mass of the galaxies in the Coma cluster? Compare this to the total gravitational mass computed above. What fraction of the total mass is in galaxies?

6. We derived the "cosmic energy equation" using Newtonian mechanics applied to a uniform universe. In this formulation, we used the fact that only the mass interior to some scale R affects the dynamics. In a sense, therefore, a cluster of galaxies can be thought of a portion of the universe that behaves like a $k = 1$ closed model. Its evolution therefore behaves similarly: it first expands with the Hubble flow, but the gravitational force from its overdensity causes the expansion to turn around, and it then recollapses into the cluster we see today.

   Calculate the density of the Coma cluster ρ_{cl} (in kg/m^3) using the previously derived mass and the given virial radius. Use this and $\rho_{cr,0}$ calculated above to compute the *overdensity* of the cluster

$$\Omega_{cl} = \frac{\rho_{cl}}{\rho_{cr,0}}.$$

   Is the overdensity of the Coma cluster indicative of a piece of a closed universe?

   What would be the radius R_L of the Coma cluster if it has the same mass, but a density equal to $\rho_{cr,0}$ (which we assume is the density of the universe for this problem)? This radius R_L is called the *Lagrangian radius*, and represents the comoving size of the patch of the Universe that collapsed to form the cluster, assuming we began long ago with a nearly uniform universe with only small density perturbations.