THE SHAPE OF PULSAR POLAR CAPS

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ABSTRACT

Rotation distorts the vacuum magnetic field of a pulsar from that of a simple dipole. The effect is particularly strong close to the light cylinder, but also affects the field close to the stellar surface. We find the shape and locus of the field lines which just close at the light cylinder. Their footpoints define the pulsar polar cap. We find this cap is asymmetric and distorted. We also find that the polar cap is *not* strongly elliptical. This result disagrees with calculations based on the non-rotating dipole field. We present our numerical results, and discuss consequences for interpretation of mean profiles and for pulsar statistics.

Subject headings: pulsars, magnetic fields

1. Introduction

The pulsar polar cap is the region mapped out on the stellar surface by the footpoints of magnetic field lines which do not close within the light cylinder (open field lines). It is bounded by the "last closed field lines", those lines which just turn around at the light cylinder and begin there to return to the star. Plasma flowing out from the polar cap is very likely to be the source of coherent, pulsed radio emission. Any effort at interpreting pulsar radio profiles must therefore be founded on an understanding the nature of this polar cap.

It was recognized early on that basic pulsar observations could be simply explained by a dipolar magnetic field centered on the star. Such a field produces a relatively small open field line region, which is consistent with the duty cycle of mean profiles. In addition, this magnetic geometry can nicely explain the observed rotation of polarization position angle across the pulse. Sturrock (1971) pointed out that the angular size of this open field line region, in an aligned dipole, is given by

$$\rho_{\rm PC} \simeq (R/r_{\rm LC})^{1/2} \tag{1}$$

where $r_{\rm LC} = c/\Omega$ is the light cylinder radius, and Ω is the angular rotation frequency. Does this extend to a tilted dipole? There were some early suggestions (*e.g.* Narayan & Vivekanand 1982) that the radio emission region is elongated in the meridional direction, but this idea was not supported with larger data sets (*e.g.* Lyne & Manchester 1988). In fact, most subsequent interpretation of pulsar radio profiles has assumed a polar cap which is circular, with a characteristic size given by equation (1) (*e.g.*, Backer 1976; Lyne & Manchester 1988; Rankin 1990 and references therein; Gil, Kijak & Seiradakis 1993 and references therein). A circular polar cap seems to be consistent with some striking trends found in the data, particularly the period-pulse width relations established by Rankin for coral and conal pulsars.

The shape of the polar cap *on the star's surface* turns out to be very sensitive to the shape of the magnetic field lines *close to the light cylinder.* If one assumes the magnetic field is that of a pure, static dipole, aligned at an angle α to the rotation axis, the resulting polar cap is not circular. Roberts & Sturrock (1972), and also Biggs (1990), found that the polar cap in an oblique pure dipole is more elliptical than circular. It has a longitudinal size given by equation (**??**), but is compressed in the latitudinal direction, by an amount up to $\sim 60\%$ in an orthogonal rotator.

Rotation and relativity change all this. A rotating dipole is in fact a time-dependent source for the magnetic field. Basic physics leads us to expect that close to, and beyond, the magnetic field must change from the static field, which is the near-field solution, to the far-field, radiative solution (in the terminology of Jackson 1975). Deutsch (1955) solved this problem, and showed that the field of a misaligned rotating dipole does, indeed, have field lines which are strongly distorted close to the light cylinder. Romani & Yadigaroglu (1995) noted that these high-altitude distortions will connect back to the stellar surface, resulting in a polar cap that is neither circular nor simply elongated.

Our contribution in this paper is to display the effect of relativistic field line distortion on the shape of the polar cap at the stellar surface, and in the radio emission region. We locate and map the last closed field lines numerically, and from this we find the exact shape of the polar cap for a rotating "pure" dipole. We find that the polar cap is much closer to circular than was previously thought, and that it is also asymmetric, with interesting distortions on the leading edge.

2. Magnetic Field of a Rotating Dipole

Deutsch (1955) found that relativity distorts the magnetic field lines of a rotating, inclined dipole, compared to those of a pure dipole. Even small inclination angles (let α be the angle between the rotation and magnetic axes) add a toroidal component to the purely poloidal dipole field. The distortions become quite dramatic for large inclination angles. Examples of the distortions can be seen in Romani & Yadigaroglu (1996), and in Higgins & Henriksen (1997), as well as the figures in this paper.

In order to study the magnetic field structure, we reproduce the Deutsch solution using a simpler analytic approach. We mix a static dipole, parallel to the rotation axis, and two oscillating ones, perpendicular to the rotation axis. (We have put the details in the Appendix.) The region well inside the light cylinder corresponds to the "near zone", or static, solutions in Jackson's (1975) terminology, and the region well outside is the "far zone", or radiation, solutions. The region around the light cylinder is the "intermediate zone", and we expect interesting field structure there.

We developed a line-tracing code to explore the solutions. We start at the star's surface, picking points with polar angle (relative to the magnetic axis) equal to ρ_{LC}, the standard polar cap radius, and uniformly spaced in azimuth around the magnetic axis. We use Runge-Kutta integration to follow a particular field line in space. We determine whether that line closes inside the light cylinder or crosses it. We then iterate on the starting polar angle (which fixes the footpoint of the line) until we have found the field line which just closes at the light cylinder. We call this the last closed field line.

In Figures 1 through 4 we show plots of the last closed field lines for four inclination angles. All simulations assume a one second rotation period; the light cylinder is at 4800 stellar radii for a typical stellar radius of 10 km. . In each figure, we show two, two-dimensional projections, and a three-dimensional view. Distances are scaled to the light cylinder radius. The first two-dimensional view is projected onto the "rotation equator", *i.e.* looking down the rotation axis. Rotation is in the counter-clockwise direction. The second two-dimensional view is projected onto the "magnetic equator", *i.e.* looking down the magnetic axis. In each three-dimensional view, the rotation axis is indicated by a vertical line. The distortions due to rotation are apparent. Even at small angles α, the field lines are twisted out of the meridional planes (compared to a pure dipole field). At larger angles, field lines are "swept back"; lines which start equally spaced around the magnetic axis become packed together at higher altitudes.

We note that this high altitude distortion and sweep-back of field lines arise in a purely vacuum solution. Thus, these effects do not depend on arguments about plasma inertia close to the light cylinder. Rather, they are simple consequences of relativity.

3. Consequences for the Polar Cap

We also use this code to explore the shape of the polar cap. In Figures 5 and 6 we show the footpoints of the last closed field lines, for a range of inclination angles. In each figure, we show the classical polar cap in a light dotted line, from equation (1), and the true polar cap, in a dark line. Rotation is to the right in each figure. We find two striking results.

First, the polar cap is not noticeably elongated. It does not deviate much from a circular shape at the classical radius $\rho_{\rm PC}$ (indicated by dotted lines in the figures), and the deviations that exist are asymmetrical in azimuth. This result disagrees with the elongated polar cap obtained with an oblique, pure dipole. We have not come up with any simple explanation. We can only note that field lines which start on the top and bottom (north and south) of the polar region extend further away from the rotation axis than they do in the nonrotating case.

Second, the polar cap is asymmetrical, both north-south and east-west. A striking discontinuity develops on the north leading side for inclination angles $\sim 40^\circ - 60^\circ$, being strongest for angles $\simeq 45^\circ$. This discontinuity is a consequence of the rotation-induced sweepback of field lines, which causes the last closed field lines to diverge at high altitudes. This effect can be seen in Figures 1 through 4.

We have investigated the nature and origin of this asymmetric shape, particularly the "glitch" at intermediate inclination angles. Our representation of the field (as given in the Appendix) allows us to separate out the pure dipole part, $\mathbf{B}_{\rm dip}$, from the total field. At small altitudes, the deviation $\Delta\mathbf{B} = \mathbf{B} - \mathbf{B}_{\rm dip}$ is small. It is easy to show that $\Delta B/B \sim O(r/r_L)$ for $r \ll r_L$. Thus, both the magnitude and the direction of the field vary little at low altitudes compared to the field of a pure dipole. In particular, there is no low-altitude anomaly at the magnetic azimuth of the glitch, despite its discontinuous appearance.

We also find that the polar cap shape does not depend on altitude, for altitudes well within the light cylinder. It is the same whether on the star's surface or at the altitude of typical radio emission (~ 10 stellar radii, typically). It may thus be more correct, formally, to refer to this shape as a "radial cross section of the polar flux tube" (to distinguish a high-altitude cross section from the physical open field line footpoints on the stellar surface). However, in the interest of conciseness, we shall continue to use the term "polar cap".

4. Discussion and Conclusions

In this paper we have revisited the question of the shape of the polar cap in a rotating dipole. We find that the polar cap is close to circular, and its scale is well approximated by the classical radius, $\rho_{\rm PC}$. Contrary to previous work which ignored the relativistic effects of rotation, we do not find an elongated polar cap. In addition, the polar cap does show interesting asymmetries, especially the "glitch" on its northeast (leading, pole-ward) side.

Will this structure have direct consequences for observations of individual pulsars? We suspect not.

The glitch is due to the high-altitude structure of the field lines. Close to the surface, the field does not deviate significantly from a pure dipole, even in the region of the glitch. Thus, we expect little signature of this structure in polarization position angle sweeps. One might anticipate an irregular plasma flow in the region of the glitch, reflecting the higher-altitude field line divergence. Flow irregularities may give rise to irregularities in the radio emission.[1] Lines of sight which pass close to this glitch – say at an impact parameter 1/3 to 1/2 of $\rho_{\rm PC}$, might be expected to result in strongly asymmetric mean profiles (with either leading or lagging emission enhanced). Separating this out from other causes of profile asymmetry seems a difficult game, however. A glance at any modern data set – for instance the 21 cm profiles taken by Rankin, Stinebring & Weisberg (1989), or the multifrequency profiles presented by Eilek, Hankins & Rankin (1998) – shows that symmetric profiles are rare. In terms of the magnetic geometry of the polar cap, nearly all radio emission seems to be asymmetric in magnetic azimuth.

Another caveat in connecting these models to observations is the question of the filling factor for radio emission. Most work on interpreting observations has implicitly assumed that the polar cap is uniformly filled, at least in magnetic azimuth. Theoretical models do not necessarily predict this, however, For instance, the steady-state models of Arons & Scharlemann (1979; also Scharlemann, Arons & Fawley 1978) find that pair formation

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[1] This statement is no more than speculation, however, until backed up by a good radio emission model.

should occur only in the "top half" of the polar cap region (the half toward the magnetic axis). If we associate pair formation with radio emission, this model predicts an elongated emission region, smaller than the classical $\rho_{\rm PC}$ size.

Thus, while a serious discussion of observational consequences must await understanding of the emission physics in this region, we doubt that dramatic consequences will result for individual pulsars. We do expect, however, that our results will impact discussions of pulsar statistics. In particular, various authors have discussed polar caps which are elongated in either the north-south direction (as suggested by Narayan & Vivekanand 1982), or the east-west direction (as predicted by an oblique, simple dipole). These elongations can in principle be detected by looking at the distribution of pulse widths, W, against polarization sweep rates. The second of these depends on the impact angle (δ, the angle between the line of sight and magnetic axis), and the first depends on the polar cap width at the particular impact angle. Recently, Biggs (1990) and McKinnon (1993) have argued that the statistics of W, δ and the inferred inclination α are more consistent with a flattened polar cap than a circular one. Their conclusions are not consistent with our result. The discrepancies may be telling us that polar caps are not uniformly filled with radio emission. Alternatively, they may be telling us that current samples of well-observed pulsars may not be unbiased in impact and inclination angles.

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A. The Magnetic Field of a Rotating Dipole

We present a new derivation of the Deutsch (1955) solution for the magnetic field of a rotating dipole. We wish to produce the field due to a dipole, with amplitude $\mathbf{m}$, which is inclined at an angle α to the rotation axis. We represent this as a sum of a static dipole, in the $\hat{\mathbf{z}}$ direction, and two out-of-phase oscillating dipoles, in the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions. That is, we superpose the field due to an static dipole, with amplitude $m\cos\alpha$, in the z direction with that due to an oscillating dipole in the x direction, with amplitude $m\sin\alpha$, and another oscillating dipole, also with amplitude $m\sin\alpha$ in the y direction, 90 degrees out of phase with the x dipole.

The field of a static dipole is, of course, (Jackson, 1975, eq. 5.56)

$$\mathbf{B}_{\mathrm{dip}}(r) = \frac{\mathbf{3}\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}}{r^3} \tag{A1}$$

The field for a time-varying magnetic dipole is given by (Jackson 9.35)

$$\mathbf{B}_{\mathrm{osc}}(\mathbf{r}) = k^2(\hat{\mathbf{r}}\times\mathbf{m})\times\hat{\mathbf{r}}\frac{e^{ikr}}{r} + [3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr} \tag{A2}$$

where $\mathbf{B}$ and $\mathbf{m}$ are understood to carry a time-dependence $e^{-i\Omega t}$, and the real part is to be taken. Here, Ω is the angular frequency of rotation, so $k = \Omega/c$ is just $1/r_L$.

We wish to produce a dipole which rotates counterclockwise as viewed from “above”. As the real part of $e^{-(i\Omega t)}$ is $\cos(\Omega t)$, we add the y oriented field with an added factor of i, giving it $\sin(\Omega t)$ dependence. However, we are really only interested in the field for a fixed time, so we replace all phase factors by the factors $\cos(r - r_*)/r_L$ and $\sin(r - r_*)/r_L$. This fixes the magnetic pole at the stellar surface to be in the direction $(\sin\alpha, 0, \cos\alpha)$, and accounts for the relatively minor time delay from the origin to the stellar surface.

After superposition of all three fields, and taking the real part of all terms, we arrive finally at the result. In Cartesian components aligned with the rotation axis, it is

$$\begin{aligned}
\mathbf{B}(\mathbf{r}) &= m\cos\alpha\mathbf{F}_1\,\frac{1}{r^5} \;+\; m\sin\alpha\;\mathbf{F}_2\,\frac{\cos[(r-r_*)/r_L]}{r^3 r_L^2} \\
&+ m\sin\alpha\;\mathbf{F}_3\frac{\sin[(r-r_*)/r_L]}{r^3 r_L^2} \\
&+ m\sin\alpha\;\mathbf{F}_4\left[\frac{\cos[(r-r_*)/r_L]}{r^5}-\frac{\sin[(r-r_*)/r_L]}{r^4 r_L}\right] \\
&+ m\sin\alpha\;\mathbf{F}_5\left[\frac{\sin[(r-r_*)/r_L]}{r^5}+\frac{\cos[(r-r_*)/r_L]}{r^4 r_L}\right] \qquad \text{(A3)}
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{F}_1 &= \left[3xz\hat{\mathbf{x}}+3yz\hat{\mathbf{y}}+(3z^2-r^2)\hat{\mathbf{z}}\right] \\
\mathbf{F}_2 &= \left[(r^2-x^2)\hat{\mathbf{x}}-xy\hat{\mathbf{y}}-xz\hat{\mathbf{z}}\right] \\
\mathbf{F}_3 &= \left[-xy\hat{\mathbf{x}}+(r^2-y^2)\hat{\mathbf{y}}-yz\hat{\mathbf{z}}\right] \\
\mathbf{F}_4 &= \left[(3x^2-r^2)\hat{\mathbf{x}}+3xy\hat{\mathbf{y}}+3xz\hat{\mathbf{z}}\right] \\
\mathbf{F}_5 &= [3xz\hat{\mathbf{x}}+3yz\hat{\mathbf{y}}] \qquad \text{(A4)}
\end{aligned}$$

The first term is that due to the static component, and the rest to purely rotating components. The pure dipole field is reproduced by

$$\mathbf{B}_{\rm dip}=\frac{m}{r^5}\left[\mathbf{F}_1\cos\alpha+\mathbf{F}_4\sin\alpha\right] \qquad \text{(A5)}$$

and the extra component added by the rotation is

$$\Delta\mathbf{B}=\mathbf{B}-\mathbf{B}_{\rm dip}. \qquad \text{(A6)}$$

Finally, we note that it is straightforward, but rather tedious, to verify that this reproduces the Deutsch (1955) solution. Deutsch expressed his solution in terms of spherical Bessel functions of the third kind, $h_1(r/r_L)$ and $h_2(r/r_L)$. The connection to our form can be made by noting the expansion of these functions in terms of $\sin(r/r_L)$ and $\cos(r/r_L)$ (*e.g.* Abramowitz & Stegun 1964).

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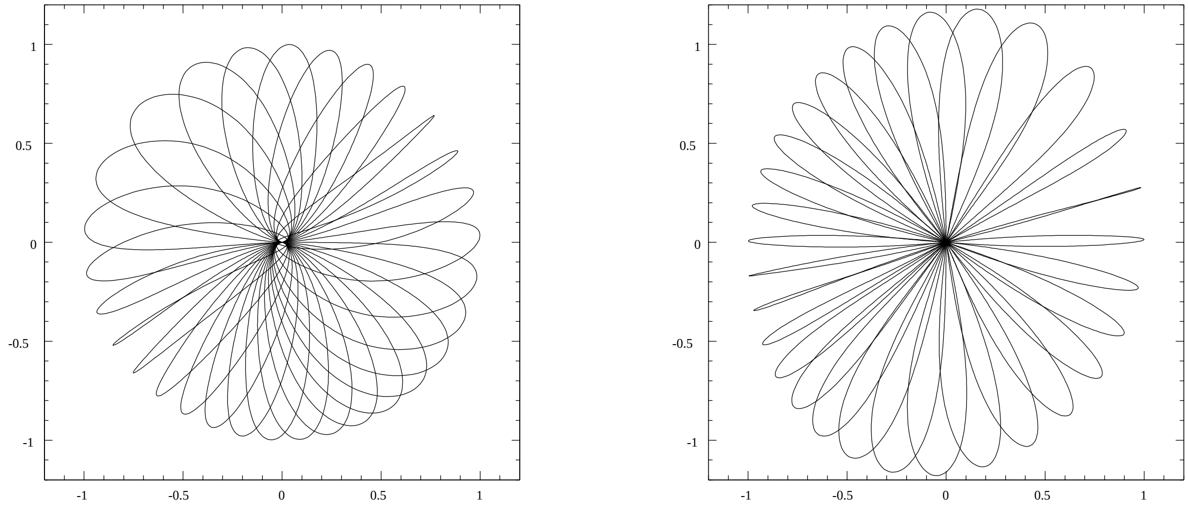

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Fig. 1.— The last closed field lines for a dipole inclined at $\alpha = 30°$ to the rotation axis. (a) A projection looking down the rotation axis; rotation is counterclockwise. (b) A projection looking down the magnetic axis. (c) a three-dimensional view, with the rotation axis indicated by the vertical line. The stellar radius is too small to be shown in these figures.

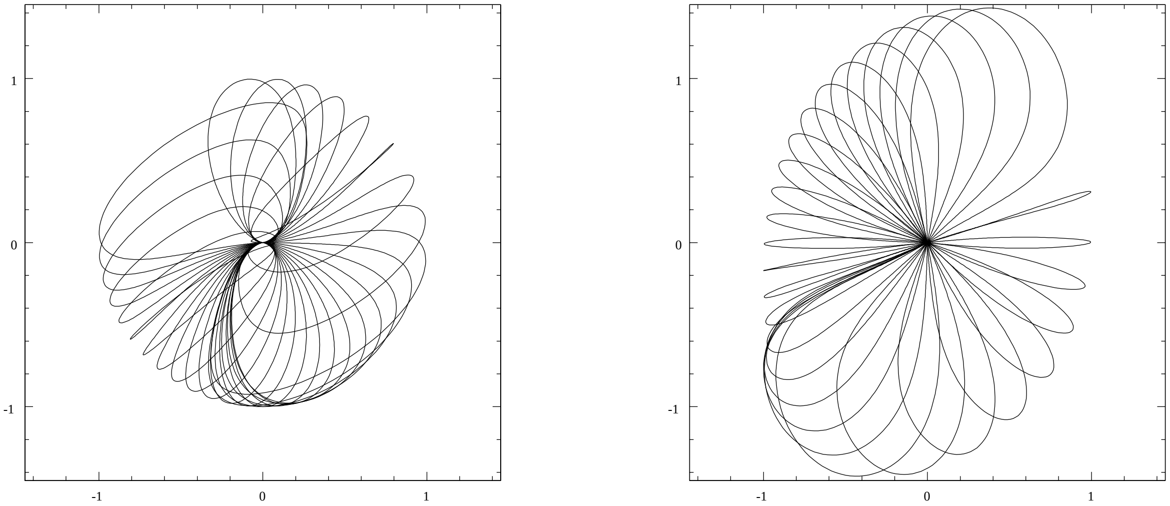

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Fig. 2.— The last closed field lines for a dipole inclined at $\alpha = 45°$ to the rotation axis. The layout is the same as in Figure 1. The "sweep-back" of the field lines, and their avoidance of one quadrant of the magnetosphere, can be seen here (all figures in this paper depict field lines which are equally spaced in magnetic longitude at the stellar surface).

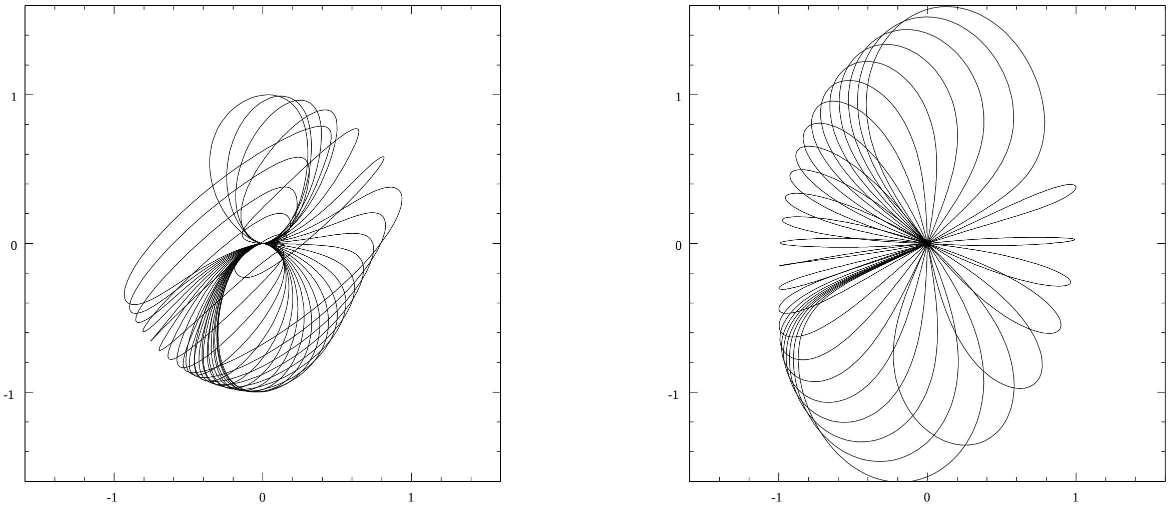

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Fig. 3.— The last closed field lines for a dipole inclined at $\alpha = 60°$ to the rotation axis. The layout is the same as in Figure 1. The distortions are quite strong at this inclination. This figure can be compared to Figure 1 of Romani & Yadigaroglu (1995), who show a 70° inclination.

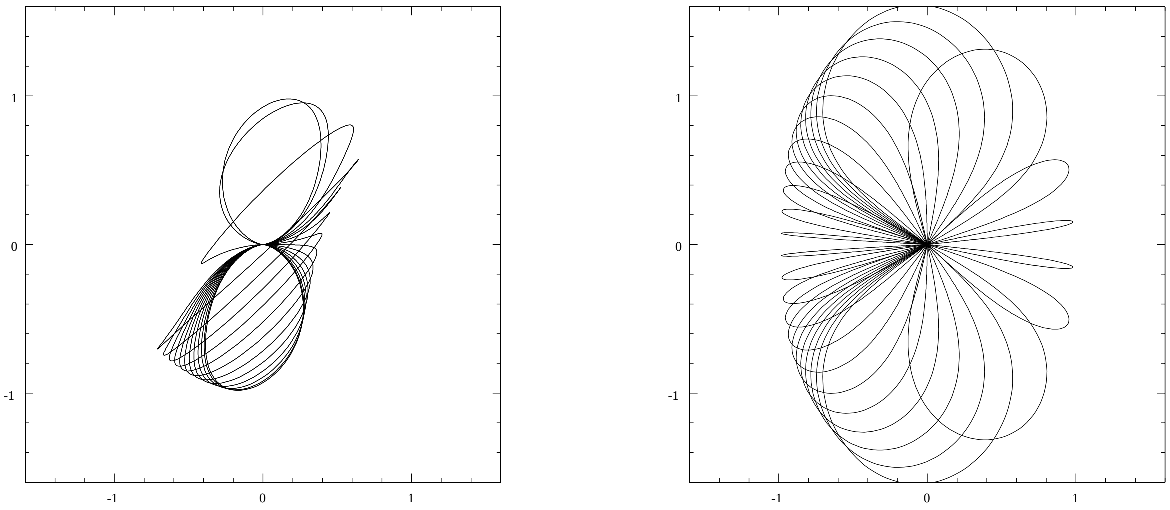

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Fig. 4.— The last closed field lines for a dipole inclined at $\alpha = 90°$ to the rotation axis. The layout is the same as in Figure 1. This figure can be compared to Figure 1 of Higgins and Henriksen (1997), who show an orthogonal rotator.

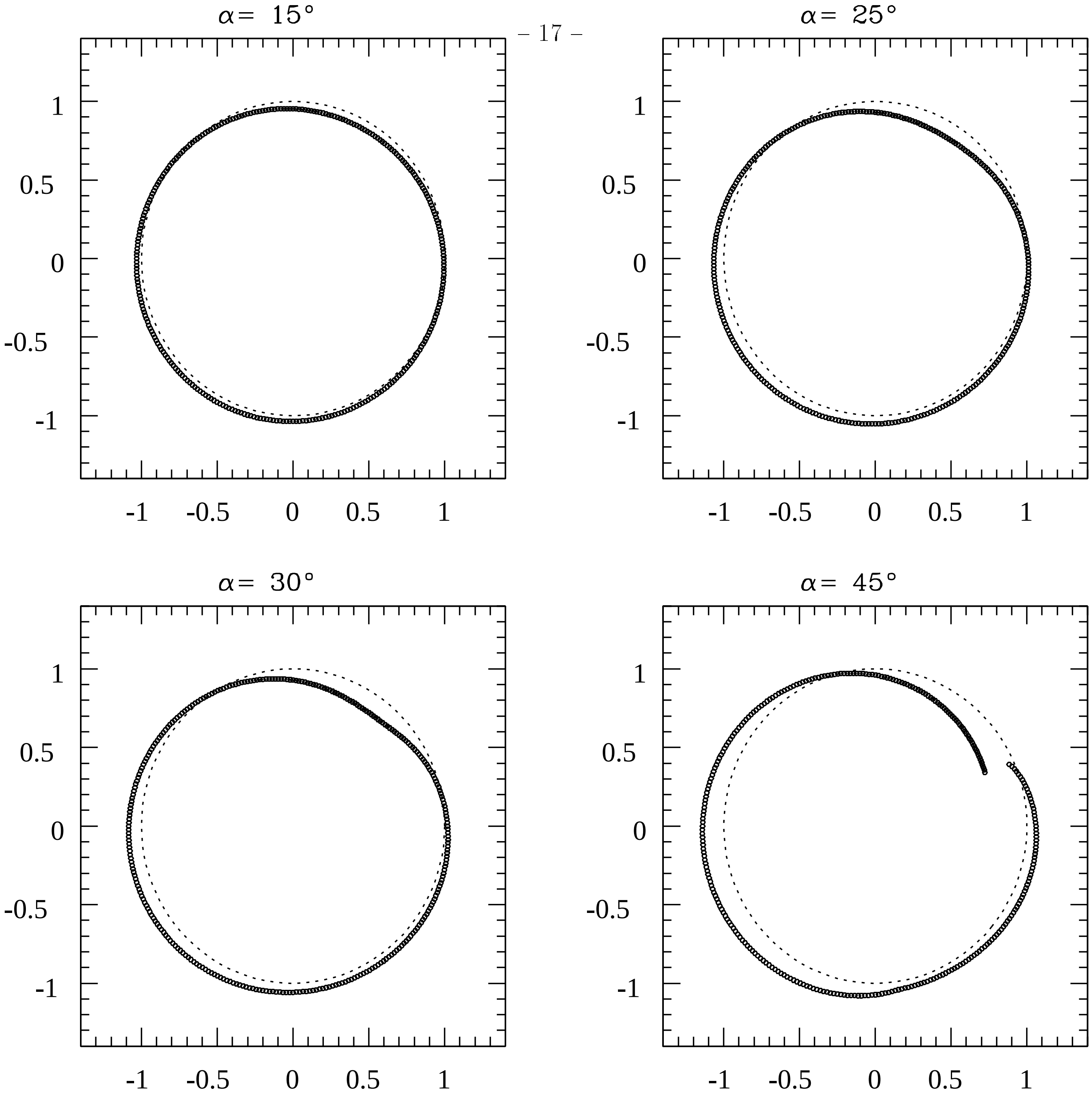

Fig. 5.— The shape of the polar cap for four low inclination angles: $\alpha = 15°, 25°, 30°$ and $45°$. The dashed line shows the classical, circular polar cap, given by ρ_{PC}. The heavy line shows the footpoints of field lines which actually close at the light cylinder. Rotation is to the right, and the magnetic axis is to the top, in these figures.

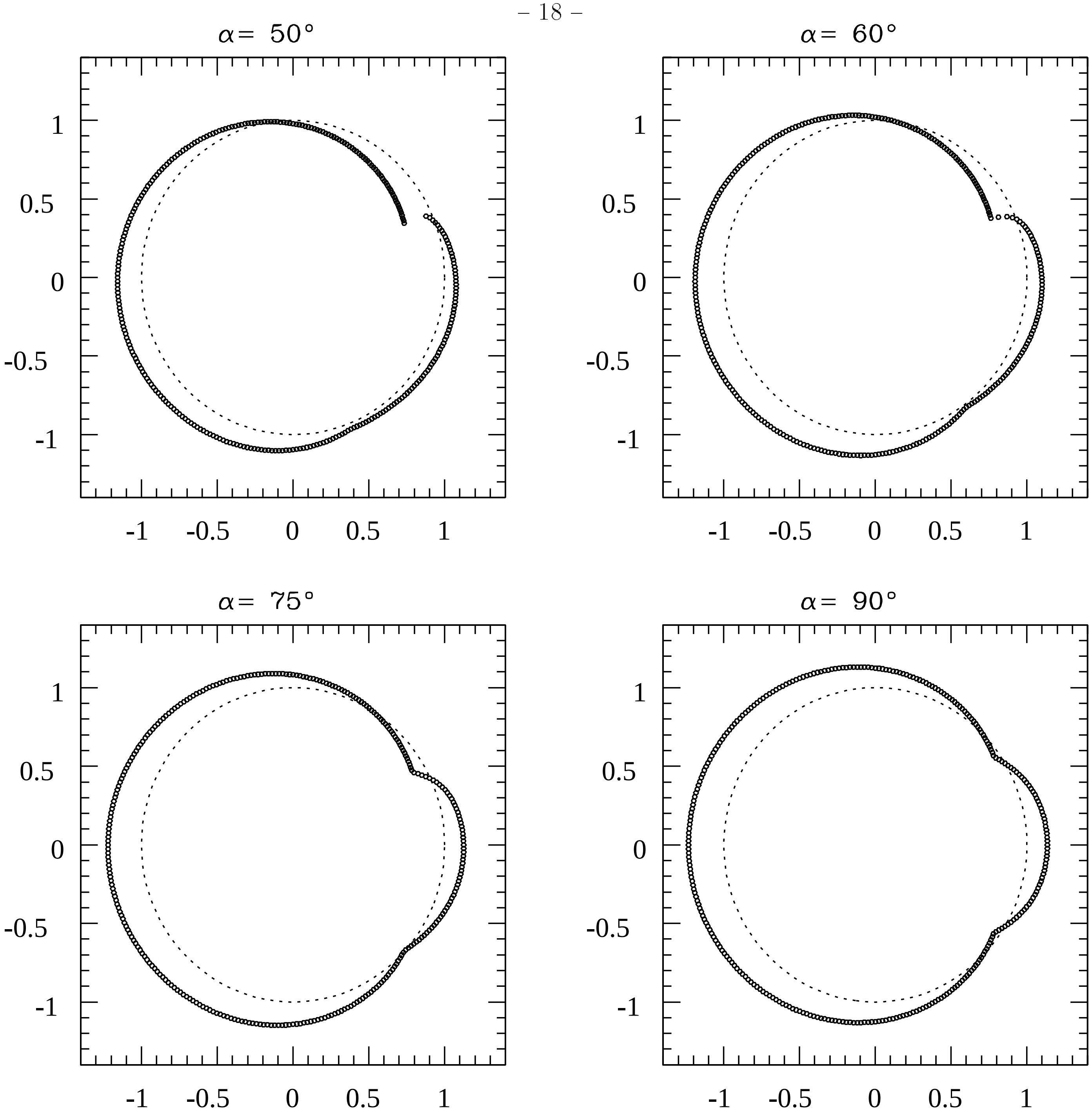

Fig. 6.— The shape of the polar cap for four large inclination angles: $\alpha = 50°, 60°, 75°$ and $90°$. The layout is the same as in Figure 5.